Proofs on Sets

Problem Set 0

- Problem Set 0 went out on Tuesday. It's due this Friday at 5:30PM.
  - Even though this just involves setting up your compiler and submitting things, please start this one early. If you start things on Friday morning, we can't help you troubleshoot Qt Creator issues!

Reading Recommendations

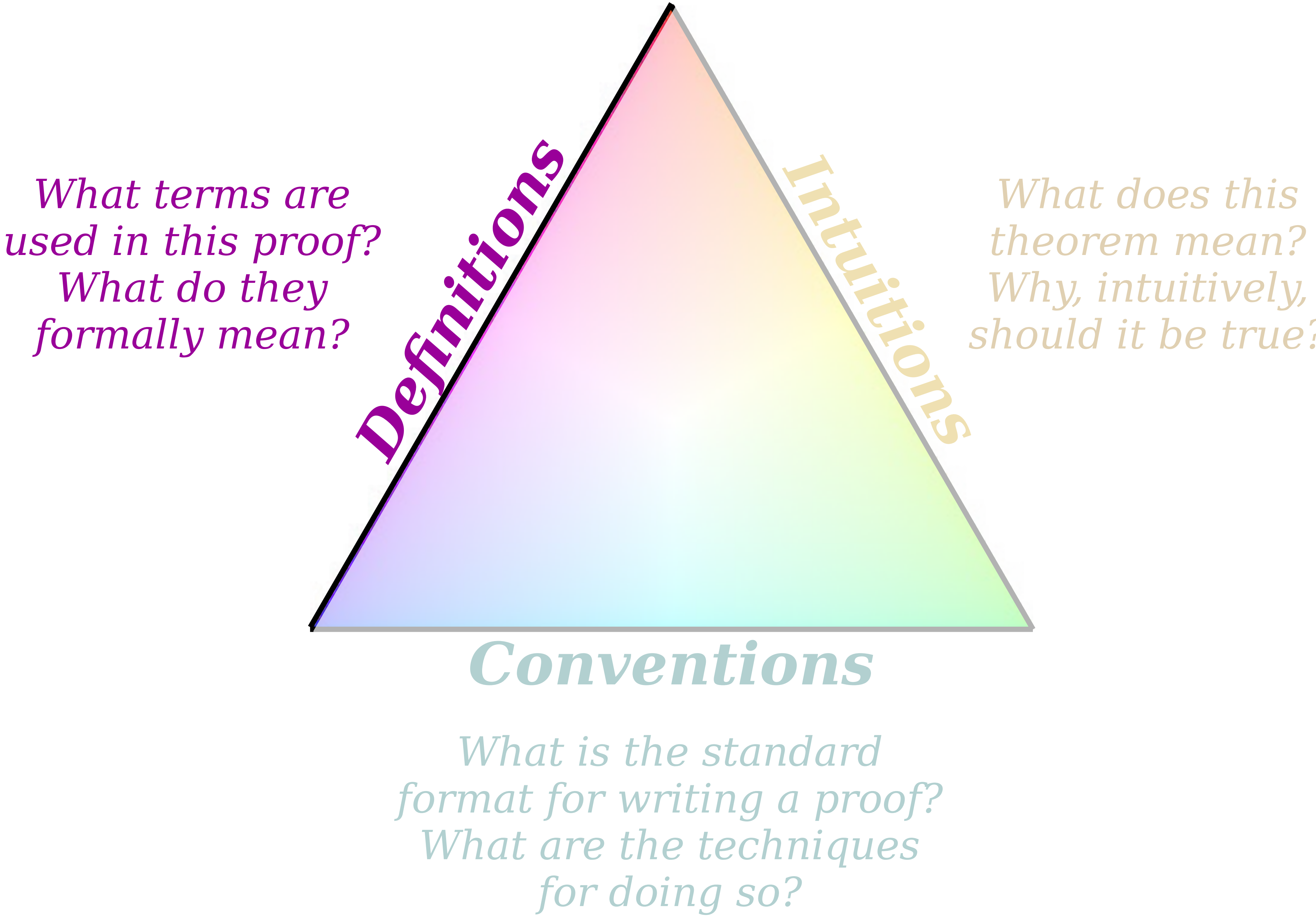
- We’ve released two handouts online that you should read over:
  - **How to Succeed in CS103**
  - **Guide to Proofs**
- Additionally, if you haven’t yet read over the **Guide to Elements and Subsets**, we’d recommend doing so.

Guide to Office Hours Released

- office hour time slots are available on the course website.
- Please read over the guide, as it talks about how our office hours work and how to make the most effective use of them.

Proofs on Sets

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.



Definitions
What terms are used in this proof? What do they formally mean?
Intuitions
What does this theorem mean? Why, intuitively, should it be true?
Conventions
What is the standard format for writing a proof? What are the techniques for doing so?

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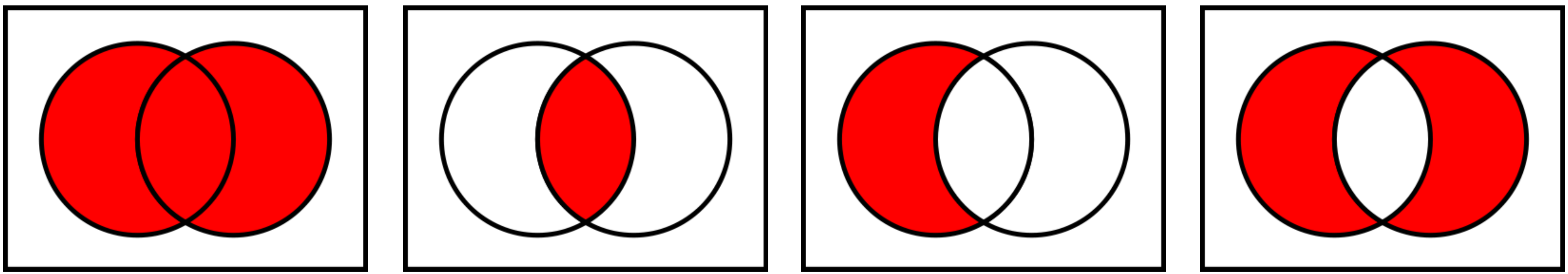
This is the ***element-of*** relation $\in$. It means that this object x is one of the items inside these sets.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

What are these, again?

Set Combinations

- In our last lecture, we saw four ways of combining sets together.



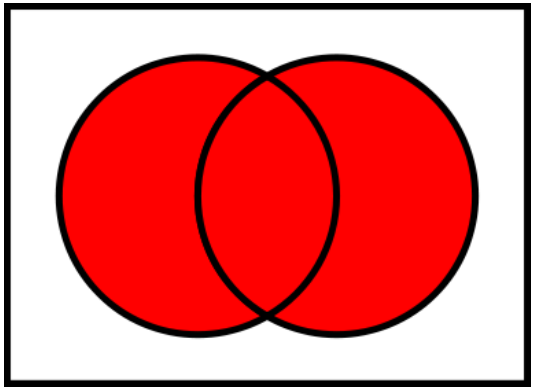
$S \cup T$ $S \cap T$ $S - T$ $S \Delta T$

- The above pictures give a holistic sense of how these operations work.
- However, mathematical proofs tend to work on sets in a different way.

Important Fact:

Proofs about sets *almost always* focus on individual elements of those sets. It's rare to talk about how collections relate to one another "in general."

Set Union



$S \cup T$

Definition: The set $S \cup T$ is the set where, for any x:

$x \in S \cup T$ when $x \in S$ or $x \in T$ (or both)

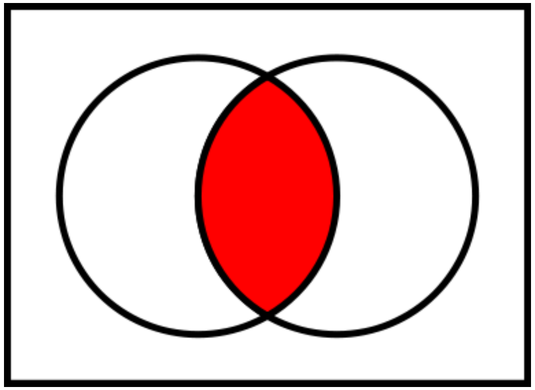
To prove that $x \in S \cup T$:

Prove either that $x \in S$ or that $x \in T$ (or both).

If you know that $x \in S \cup T$:

You can conclude that $x \in S$ or that $x \in T$ (or both).

Set Intersection



$S \cap T$

Definition: The set $S \cap T$ is the set where, for any x:

$$x \in S \cap T \quad \text{when} \quad x \in S \text{ and } x \in T$$

To prove that $x \in S \cap T$:

Prove both that $x \in S$ and that $x \in T$.

If you know that $x \in S \cap T$:

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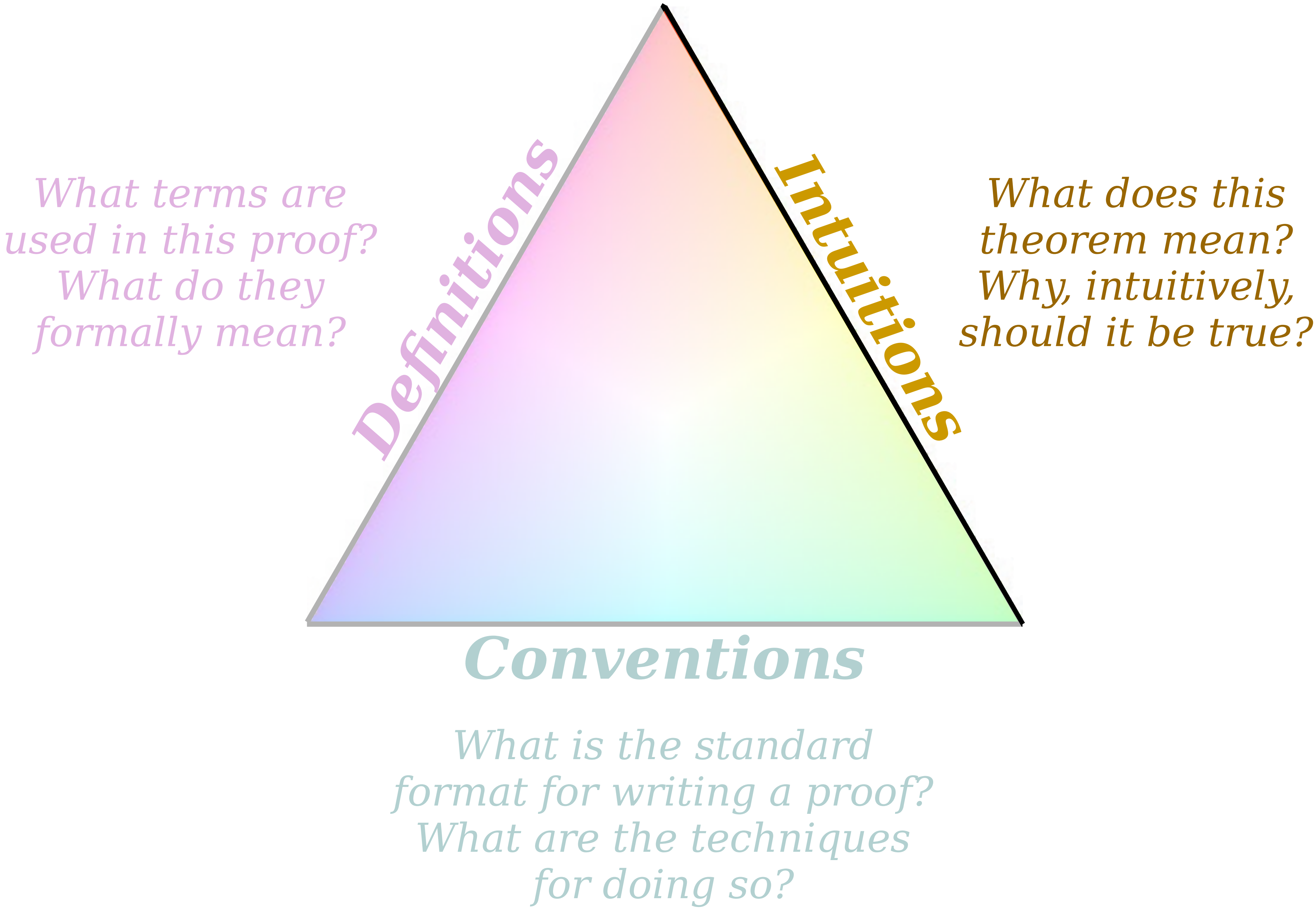
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Let's Try Some Examples!

$A = \{1, 2, 3\}$
$B = \{2, 3, 4\}$
$C = \{3, 4, 5\}$

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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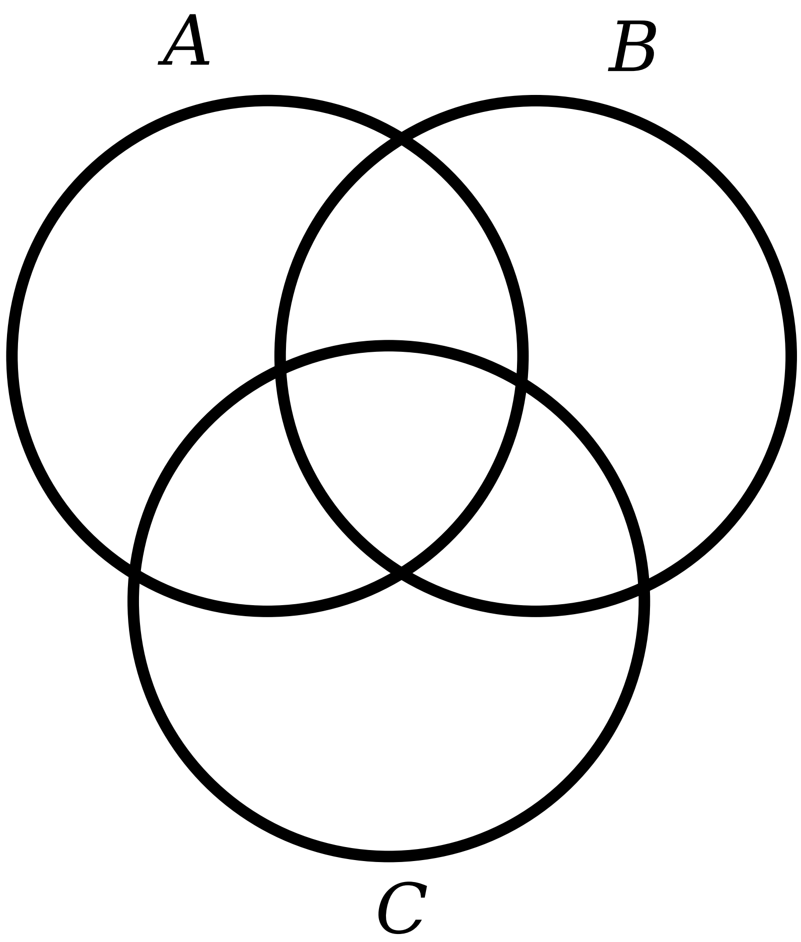

Question: Pick $x = 1$.
Is $x \in (A \cap B) \cup C$?
Is $x \in (A \cup C) \cap (B \cup C)$?

Now pick $x = 2$.
Is $x \in (A \cap B) \cup C$?
Is $x \in (A \cup C) \cap (B \cup C)$?

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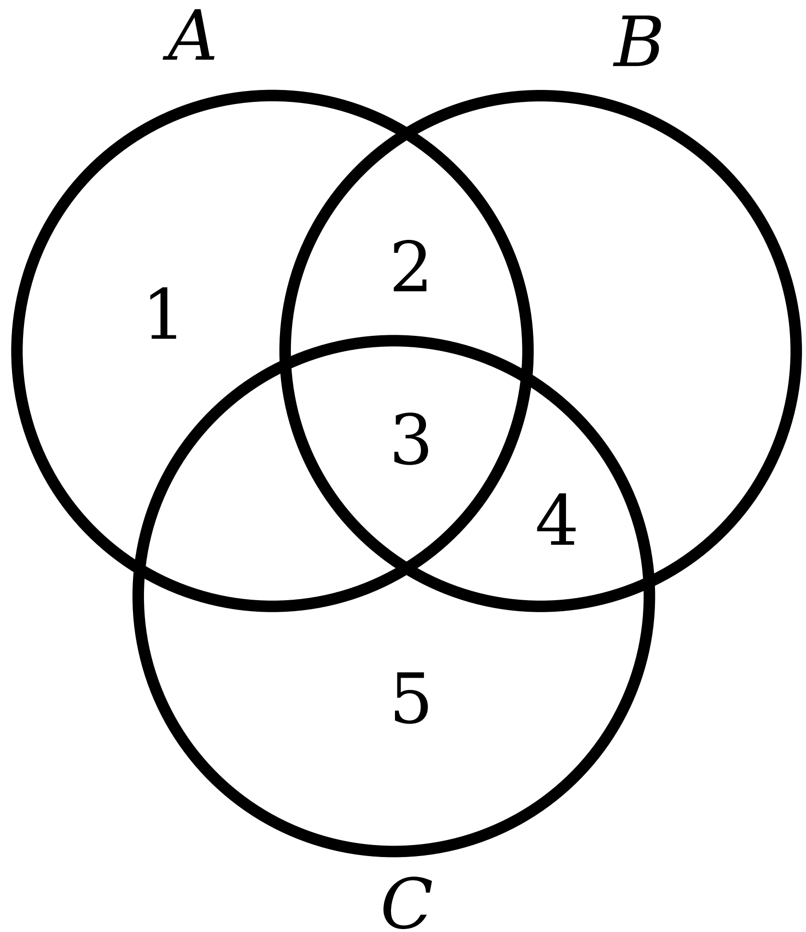
$x = 1$

✔ ✘ ✘

Is $x \in (A \cap B) \cup C$?

✔ ✘ ✘ ✘

Is $x \in (A \cup C) \cap (B \cup C)$?



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$x = 2$

✔ ✔ ✘

Is $x \in (A \cap B) \cup C$?

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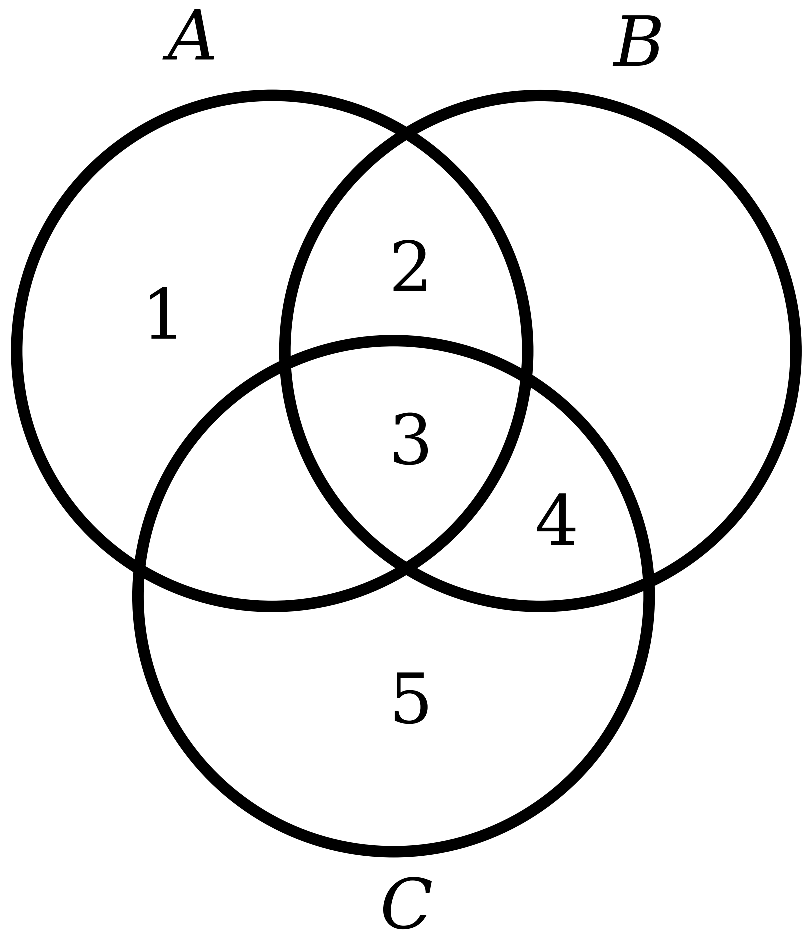
Is $x \in (A \cup C) \cap (B \cup C)$?



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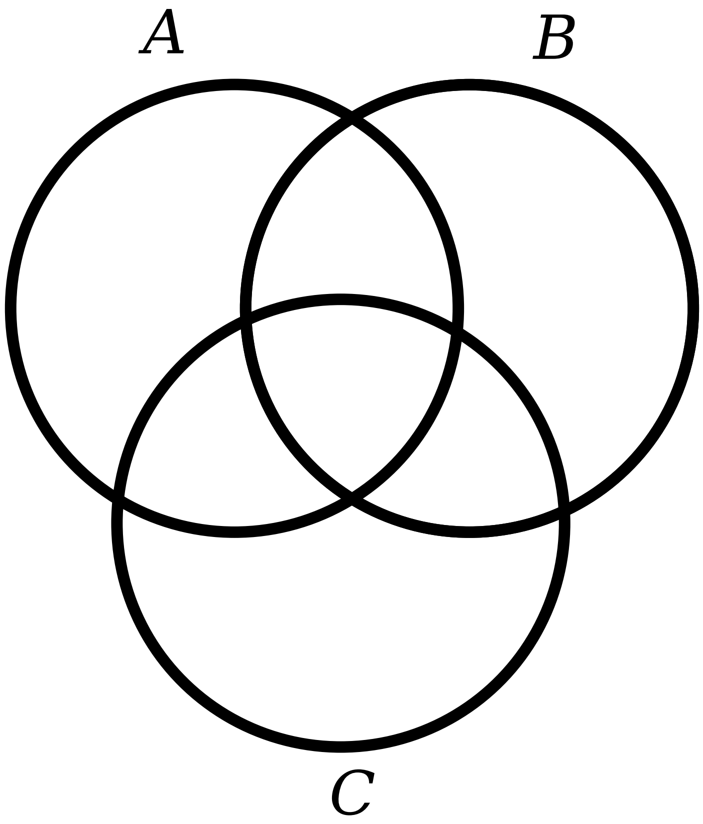

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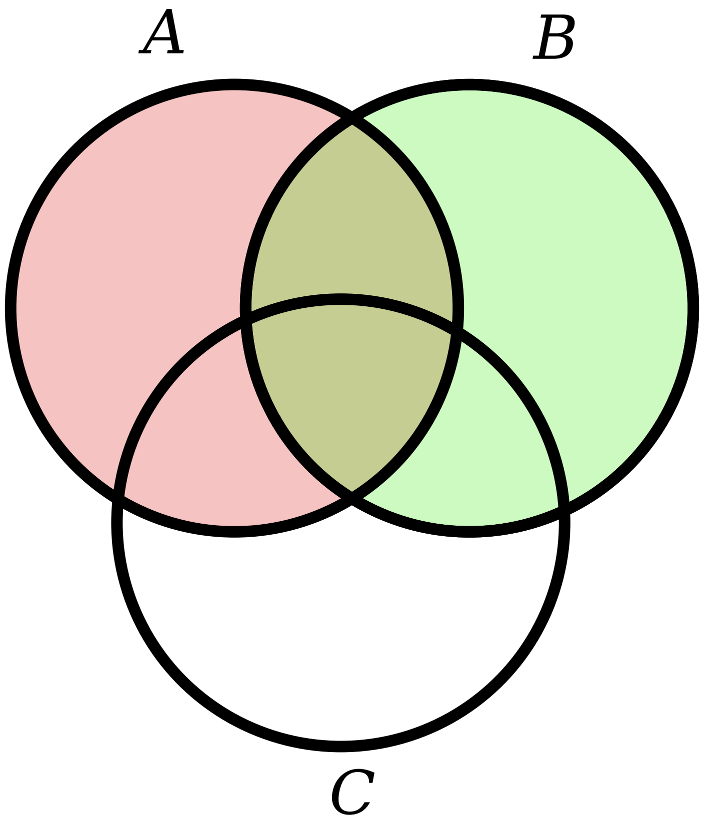

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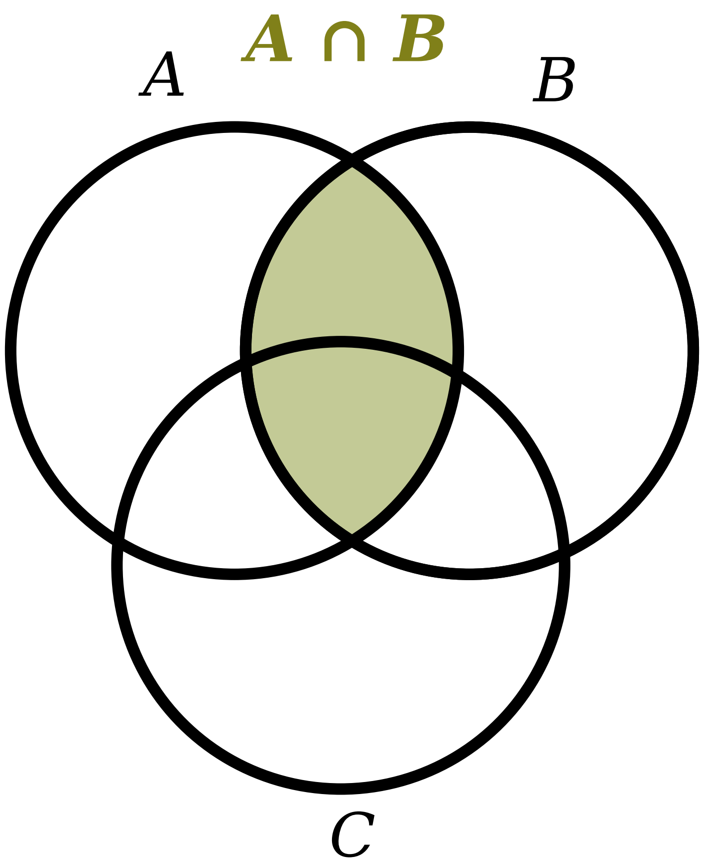

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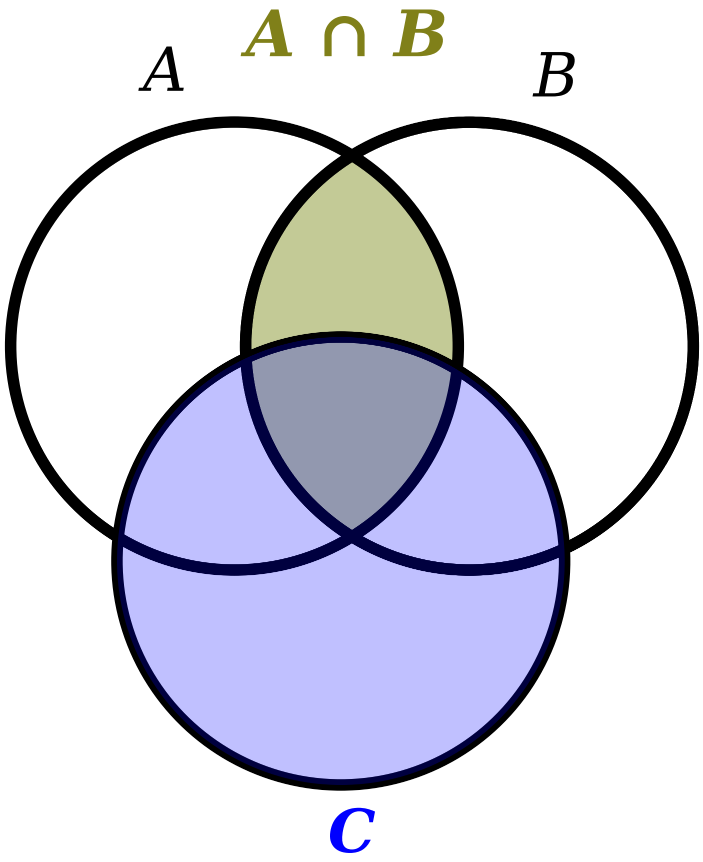

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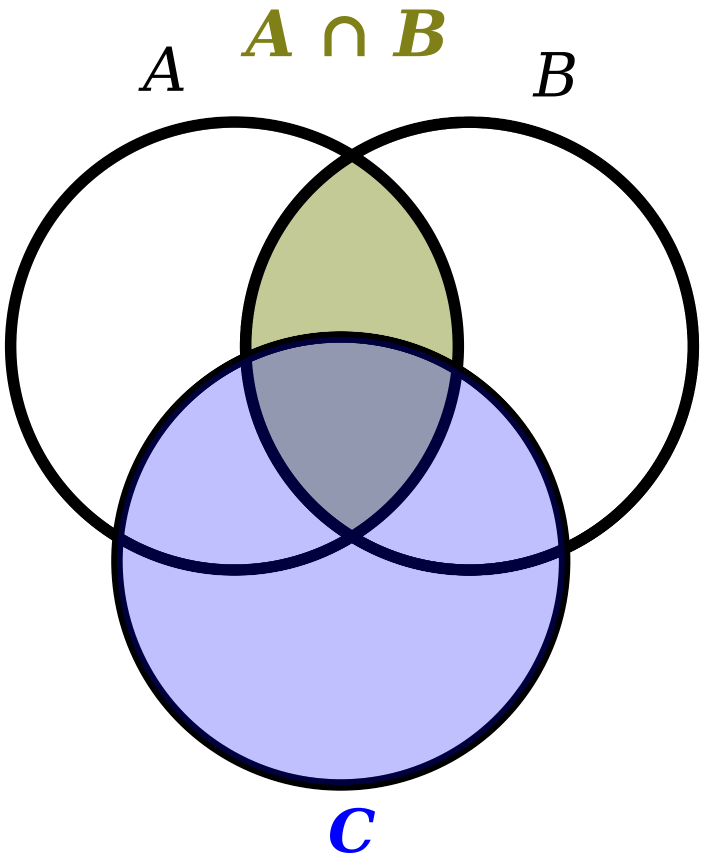

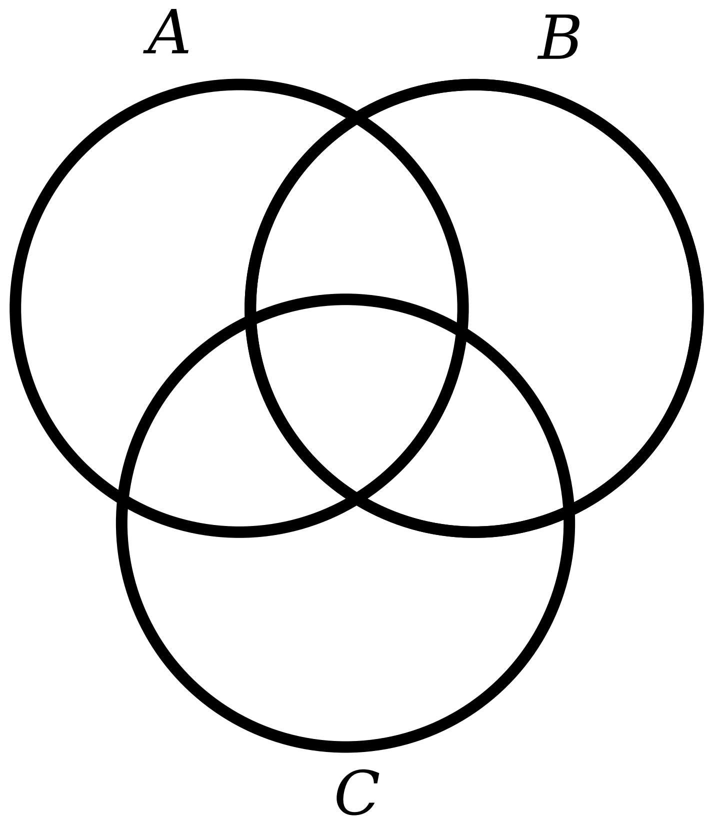

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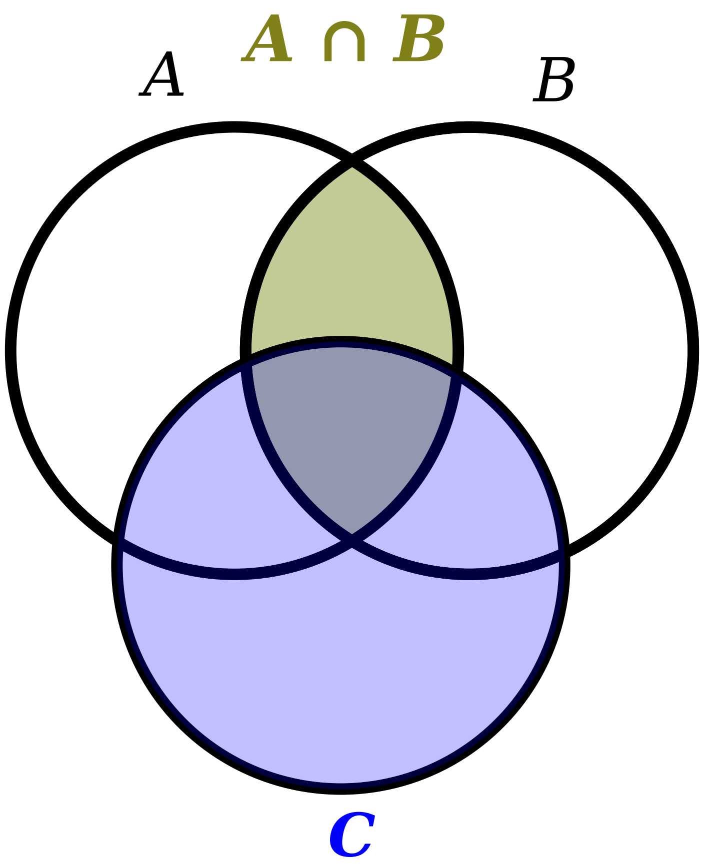

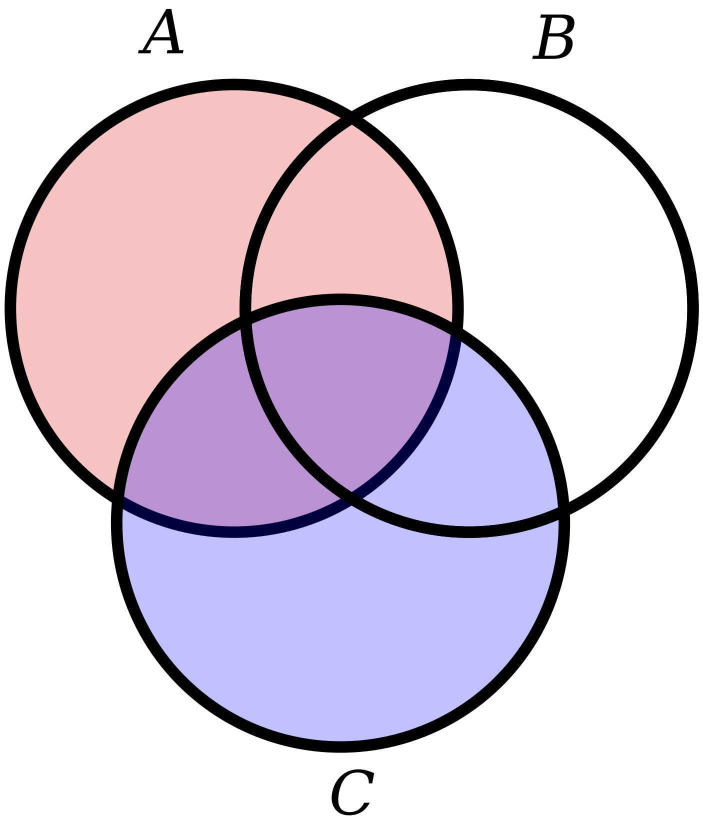

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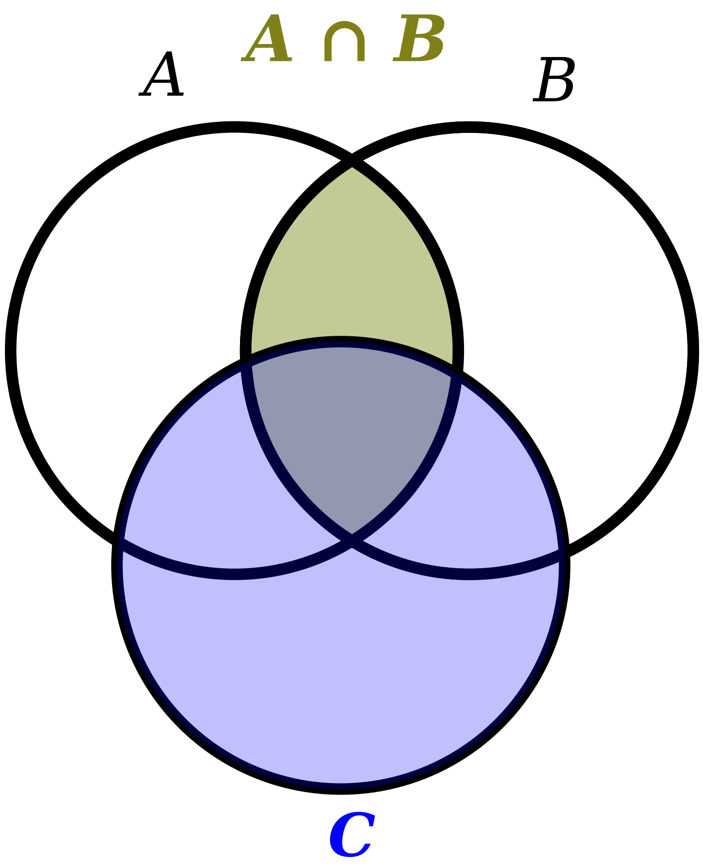

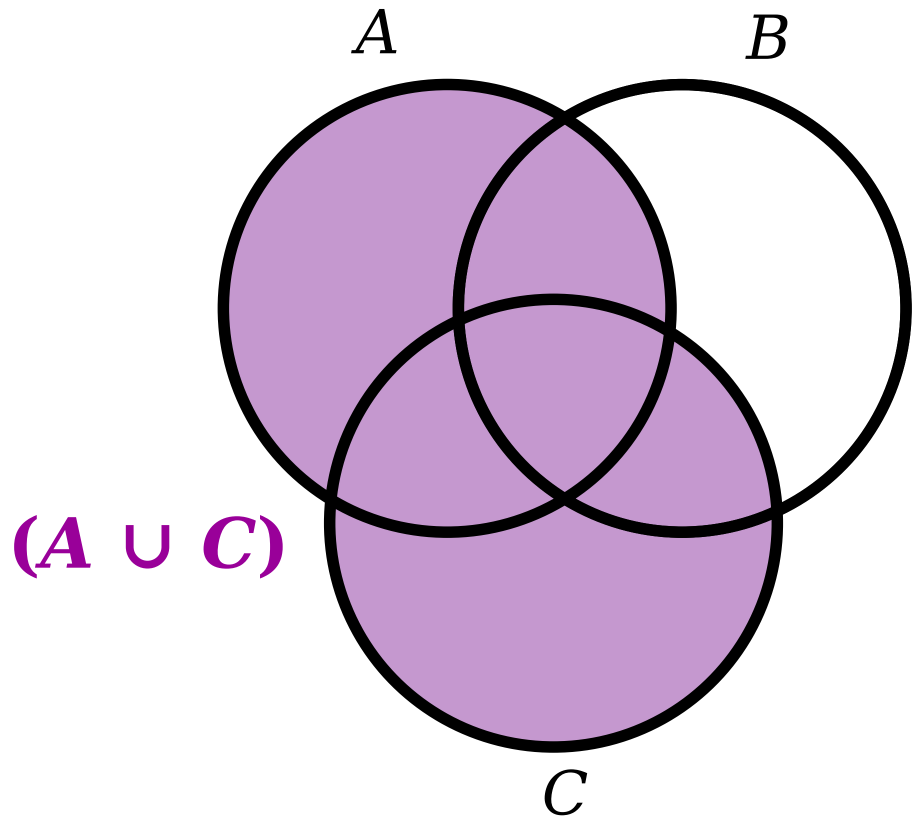

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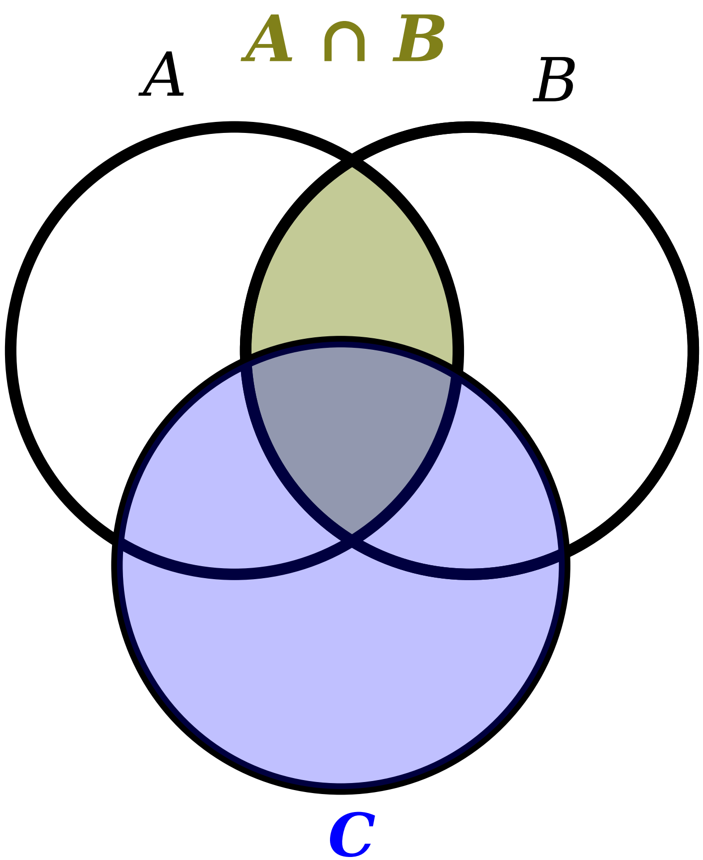

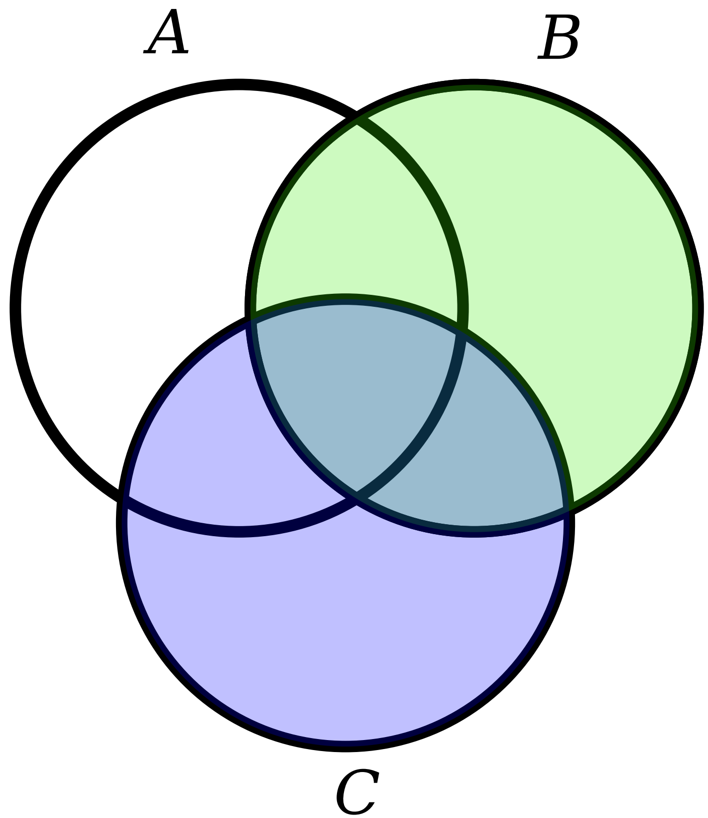

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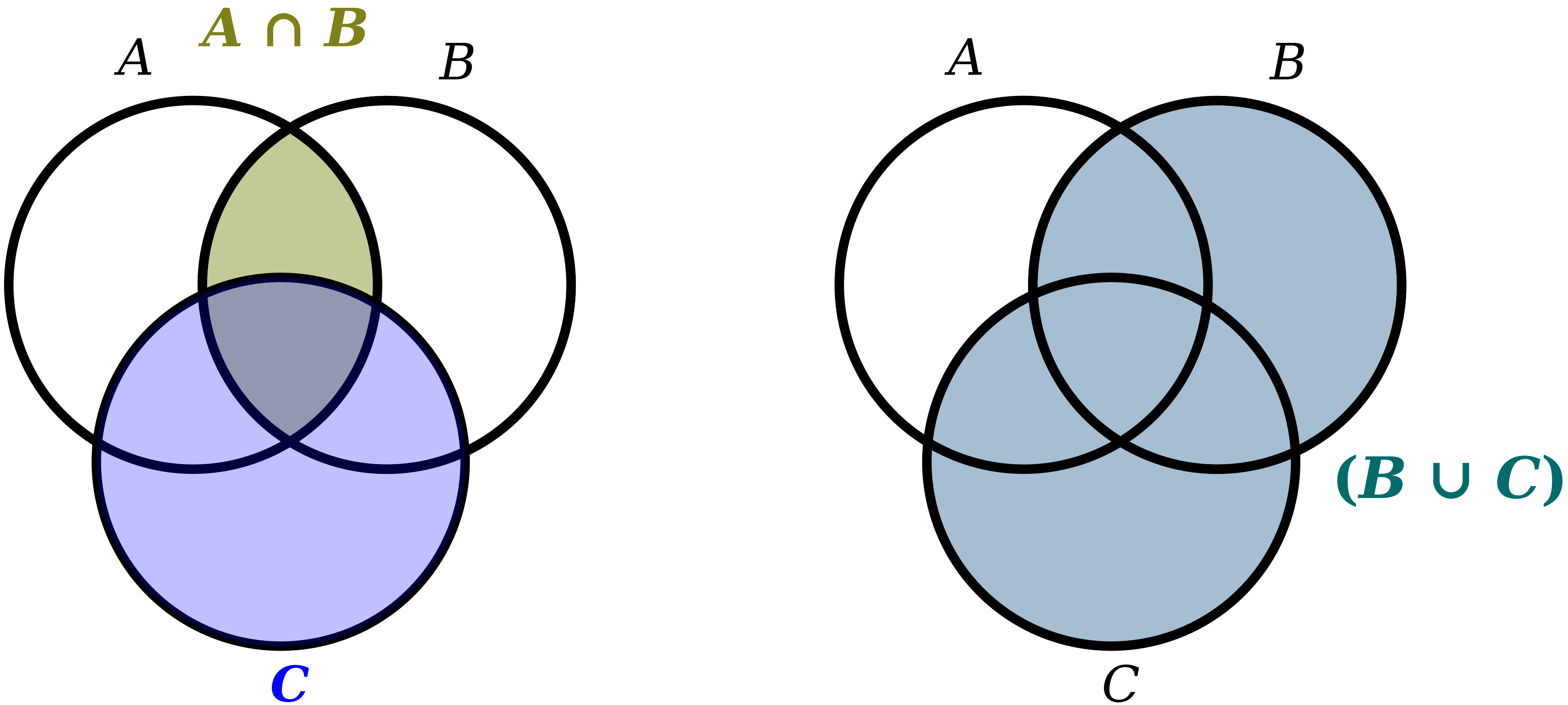

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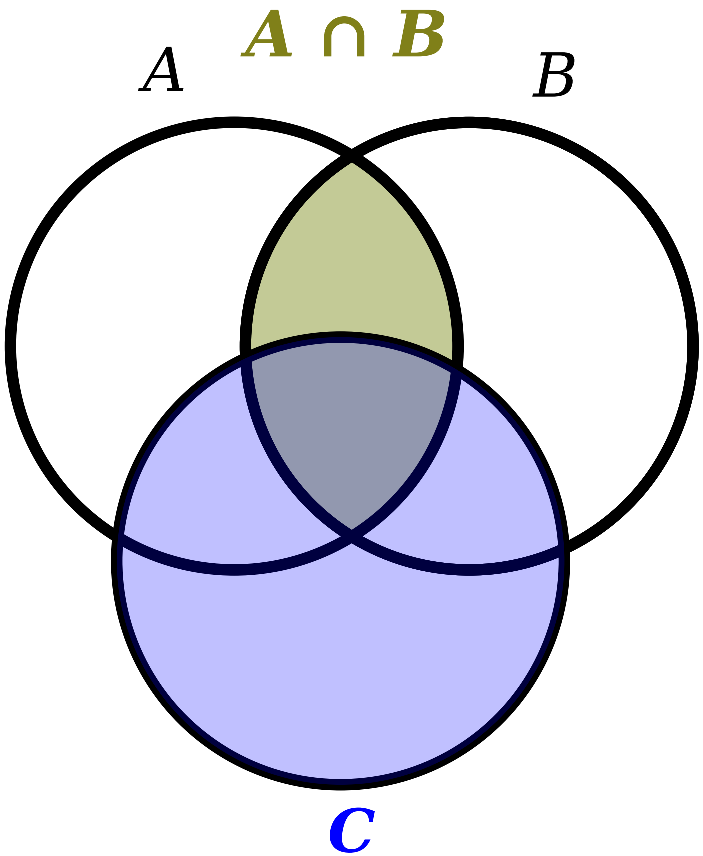

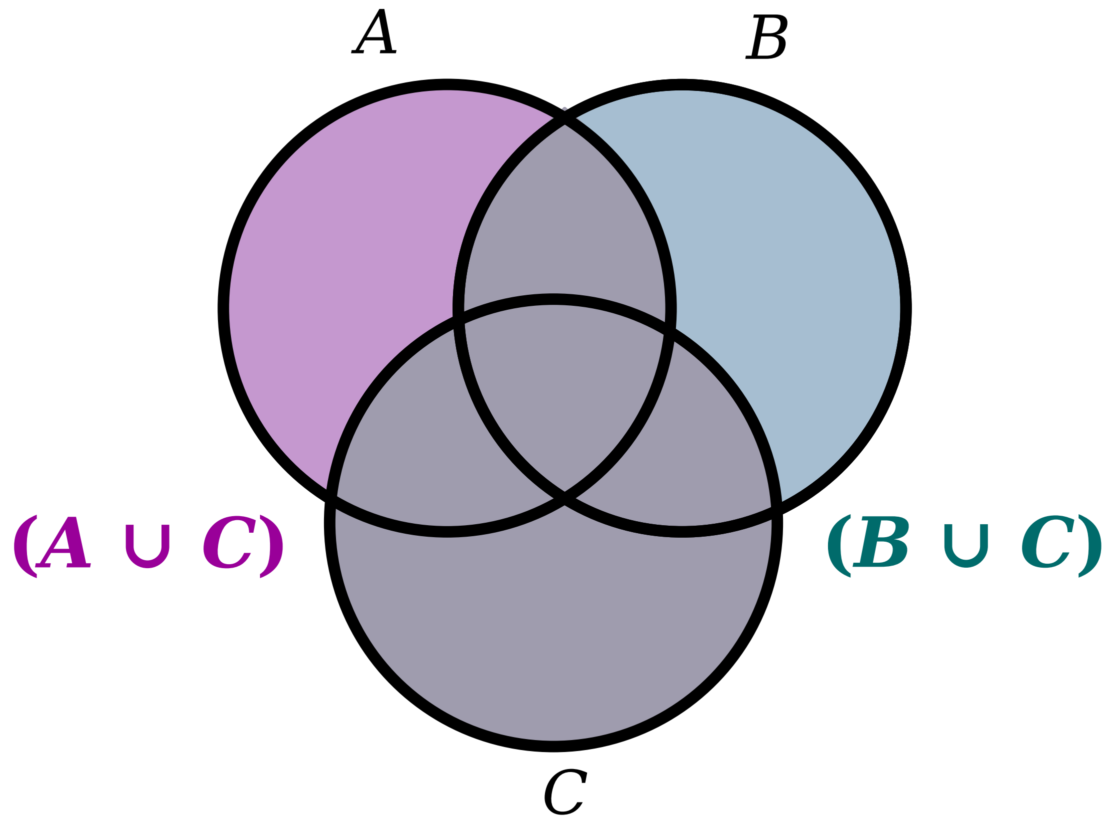

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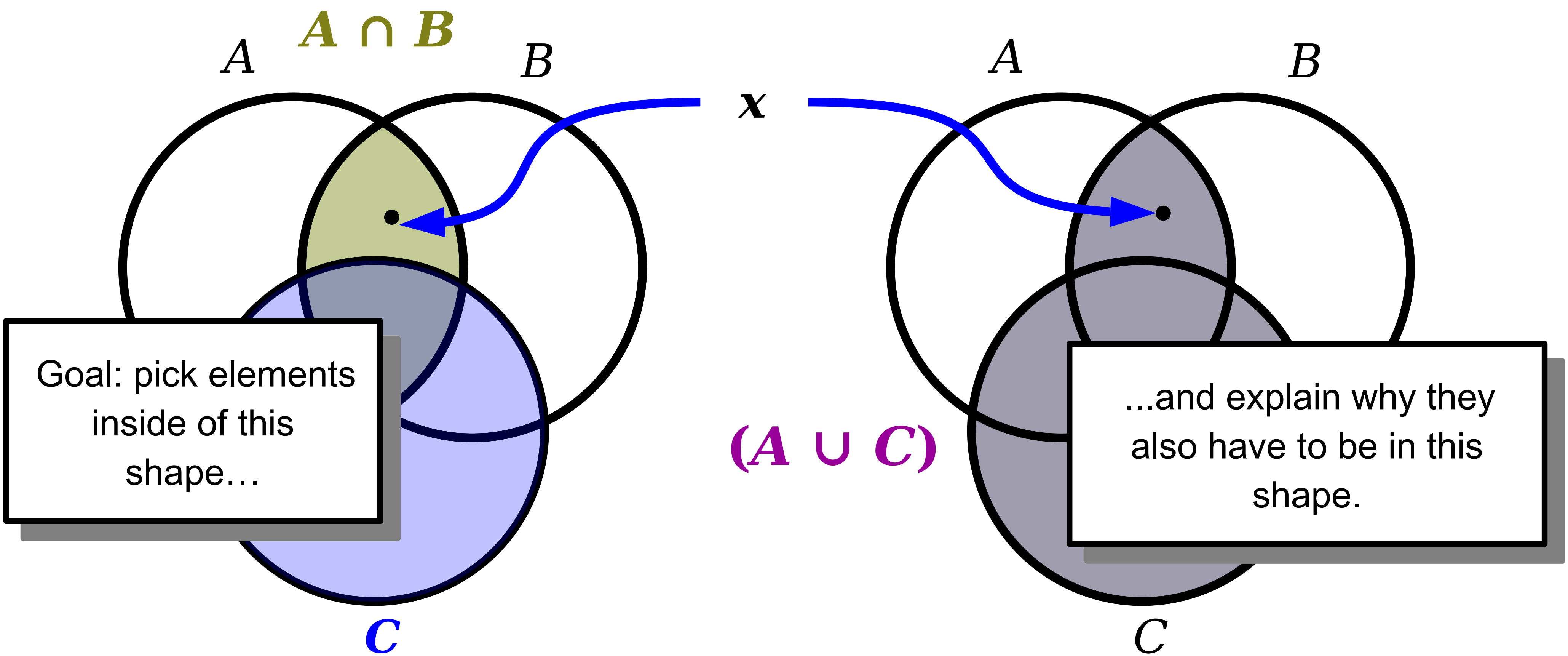

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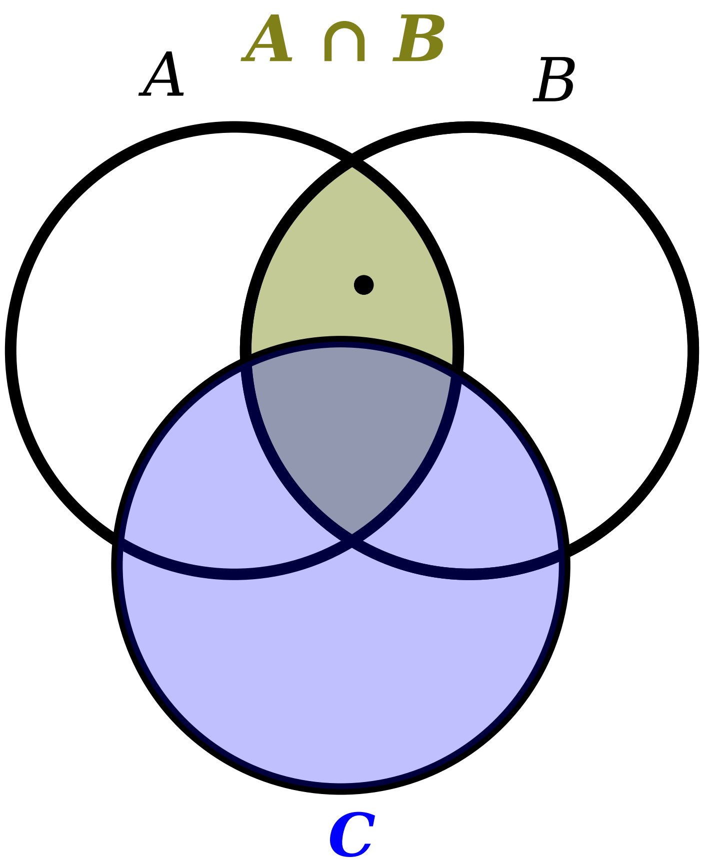
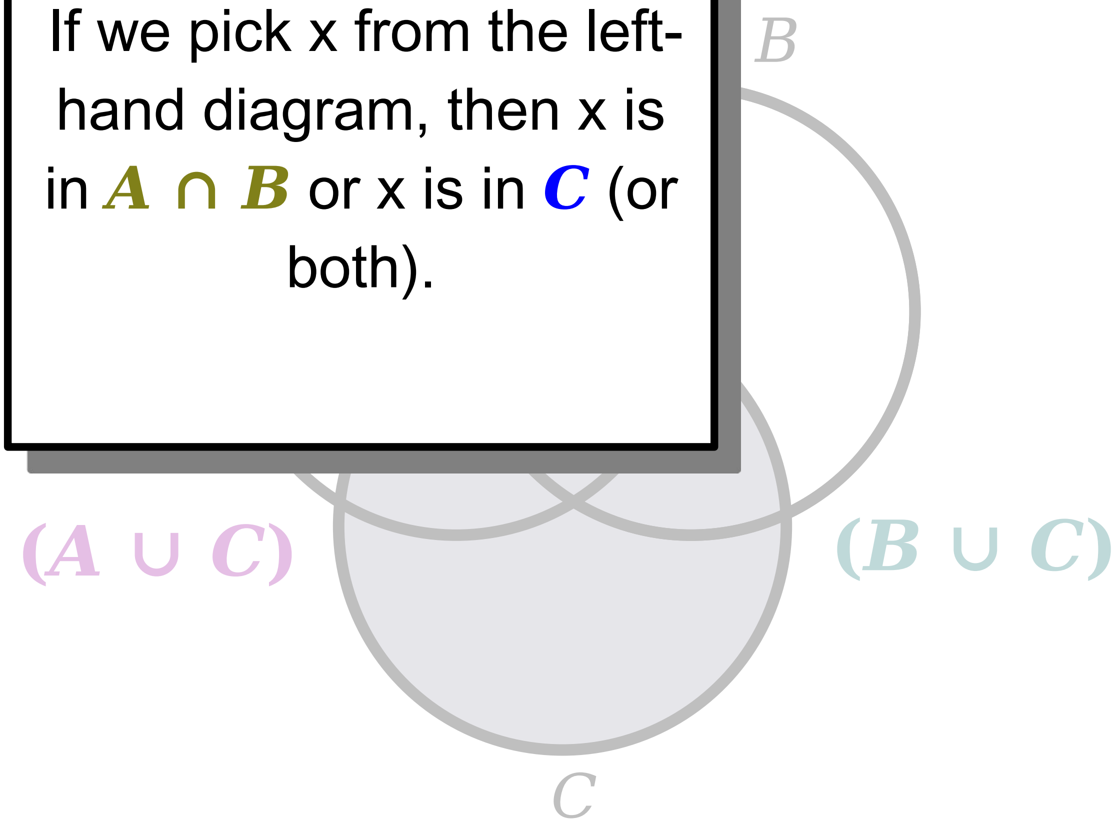


If we pick x from the left-hand diagram, then x is in ***A* ∩ *B*** or x is in ***C*** (or both).



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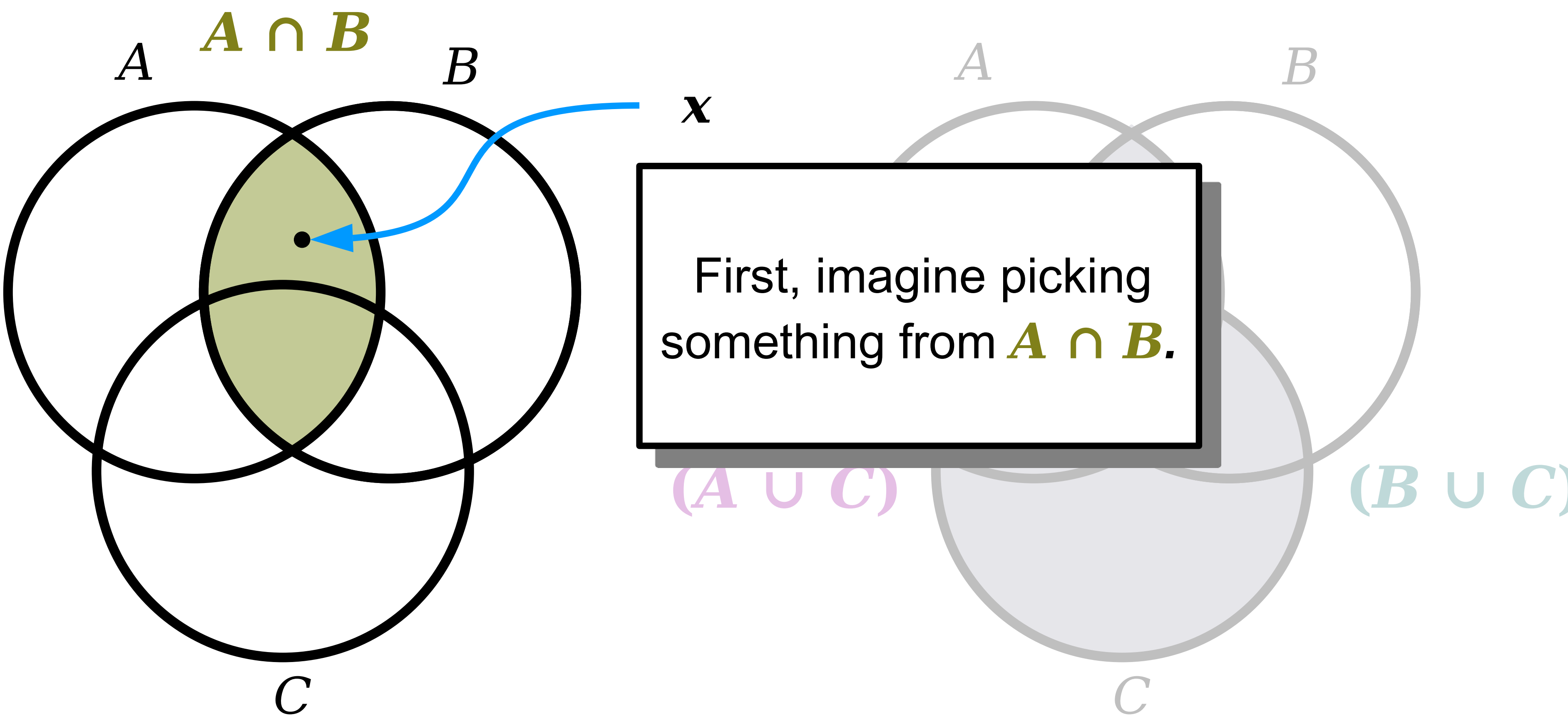

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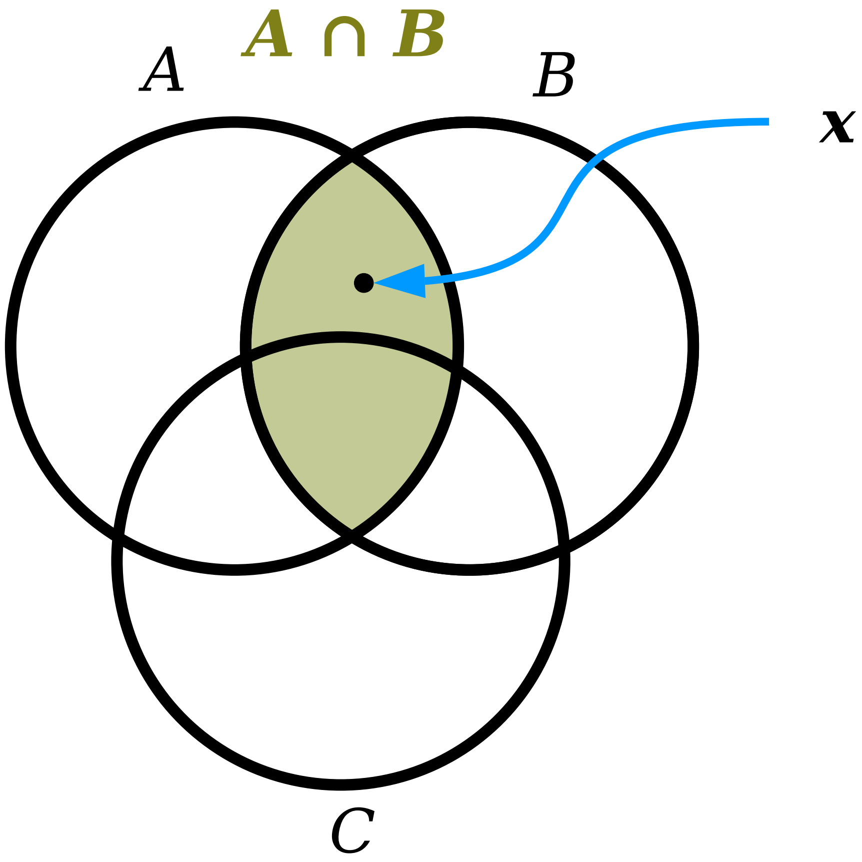

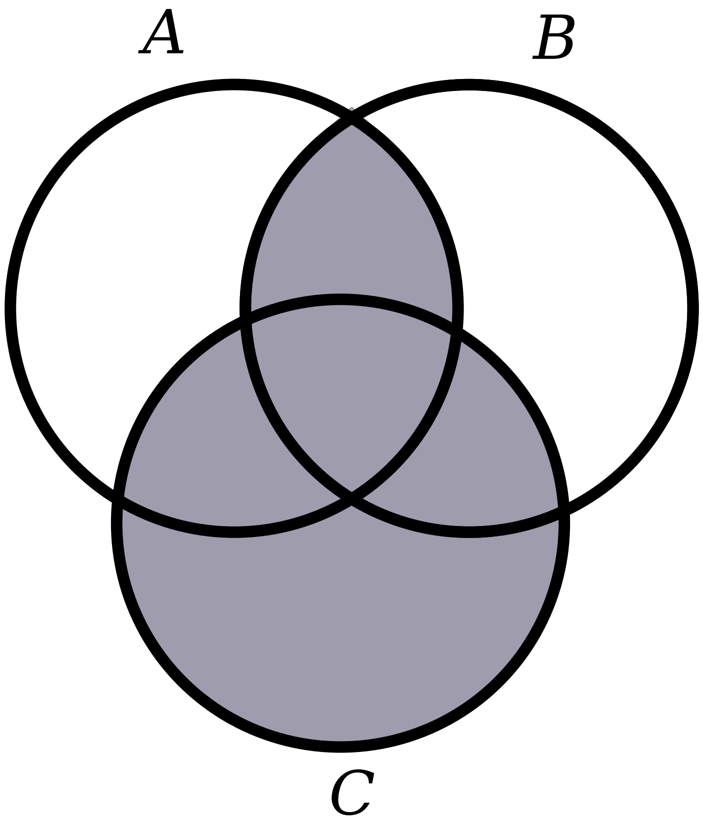

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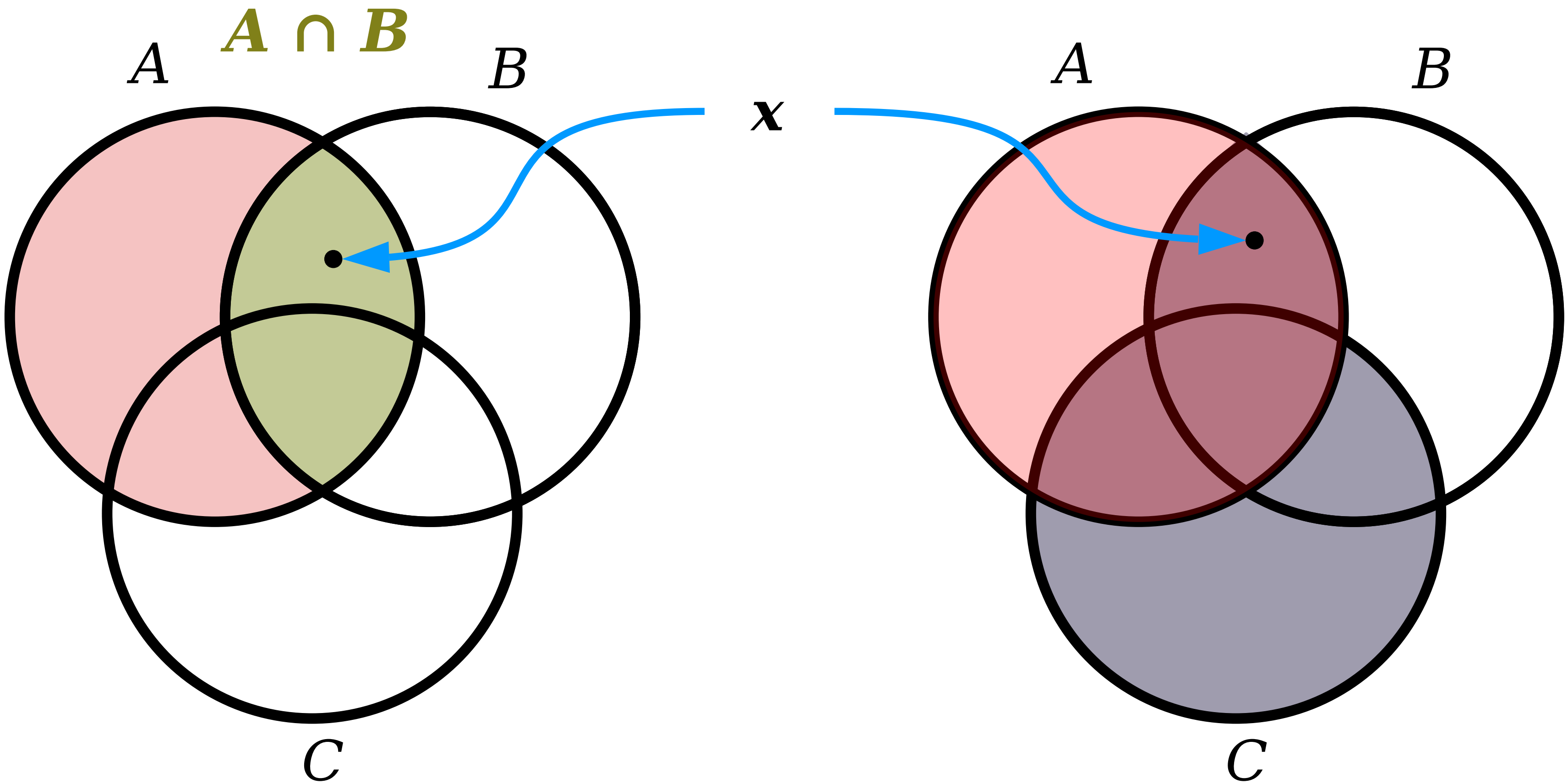

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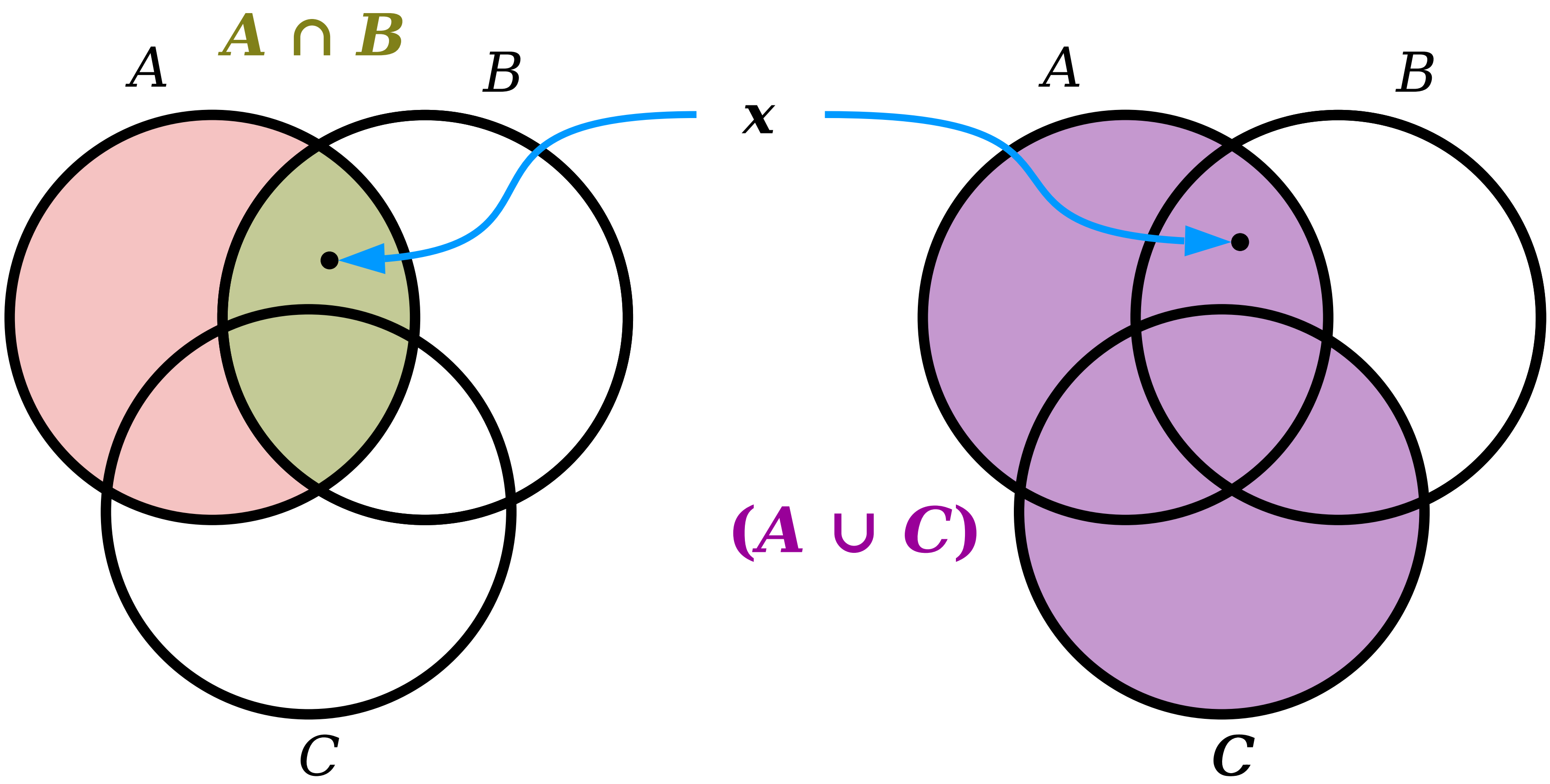

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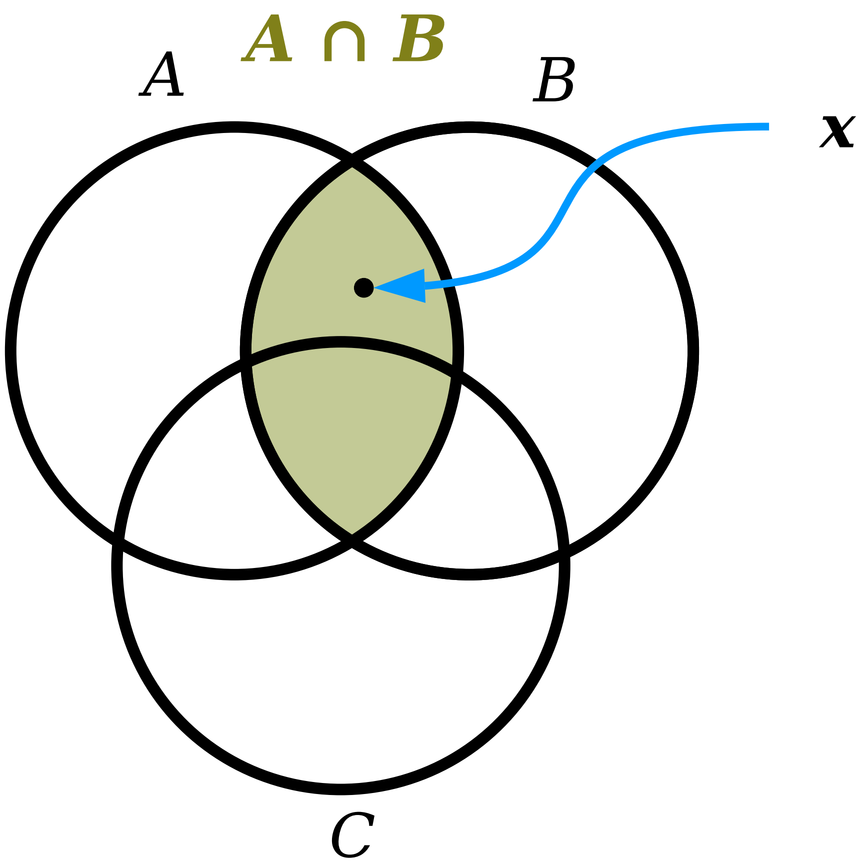

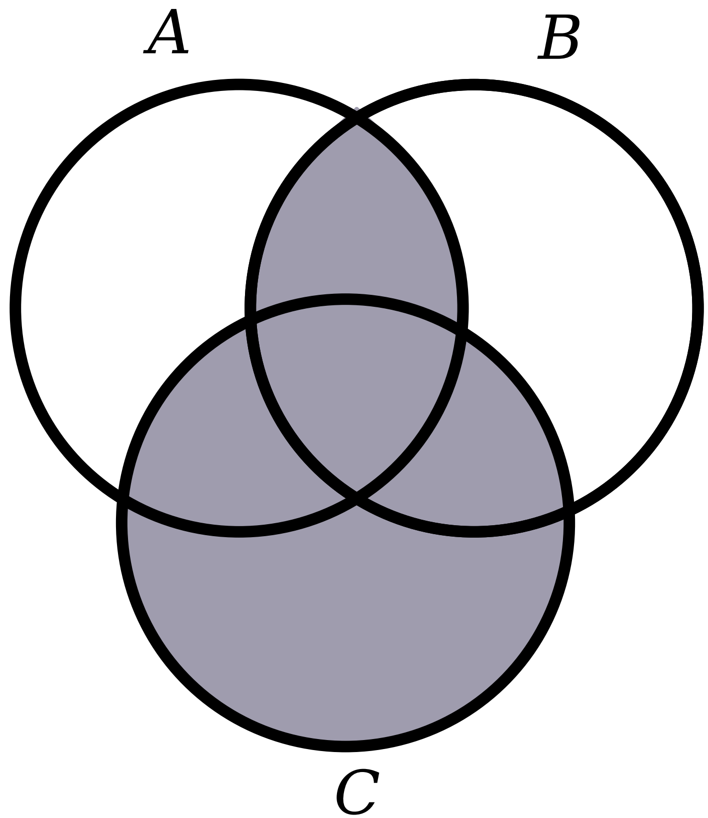

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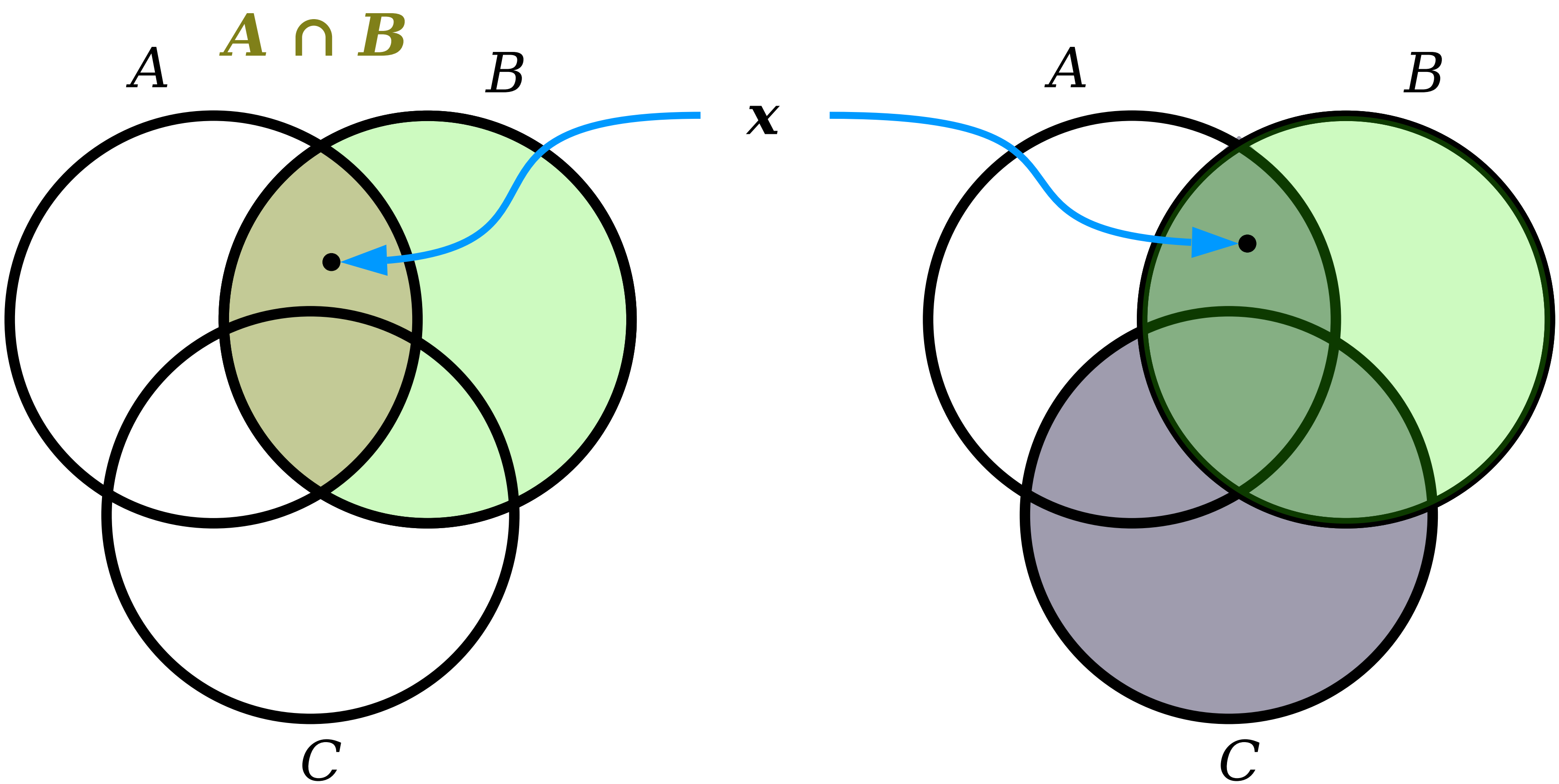

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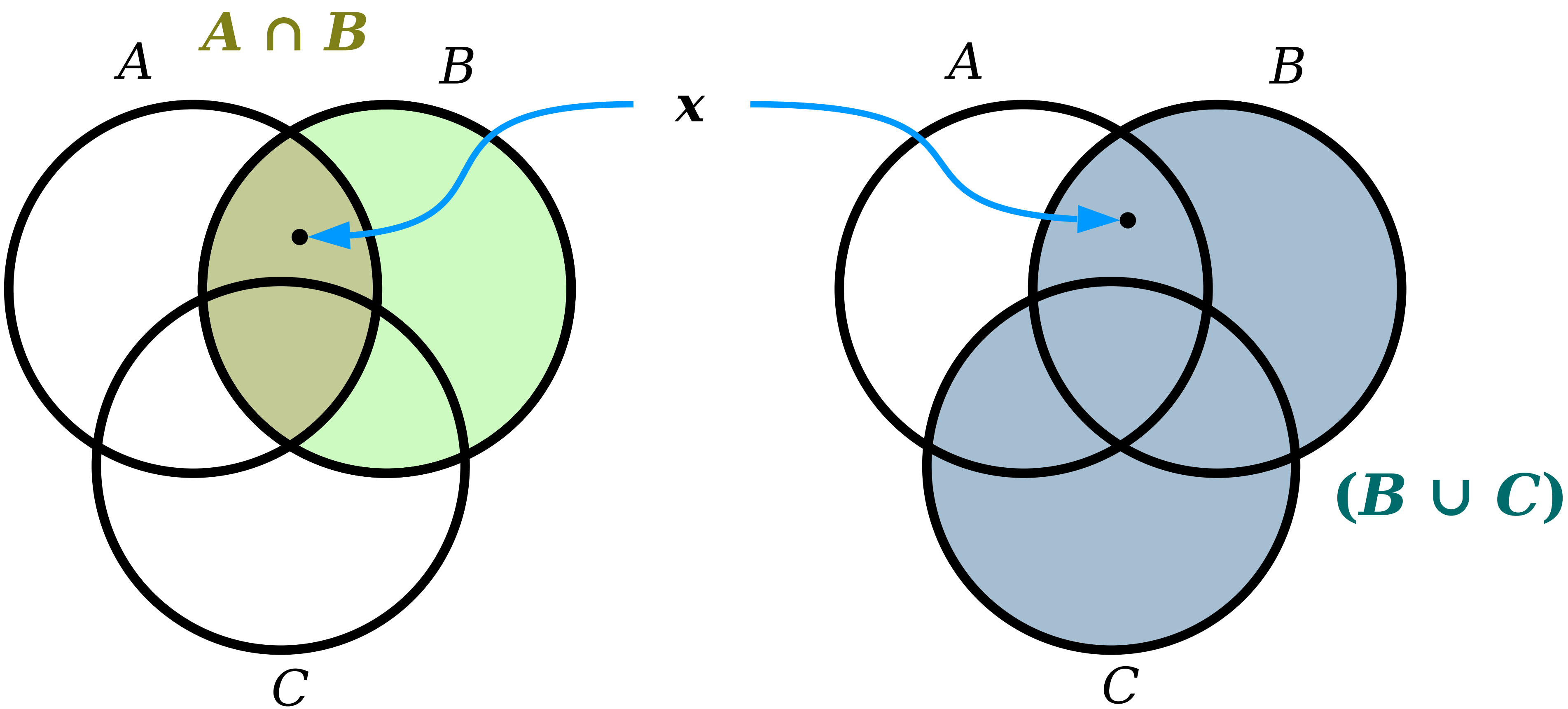

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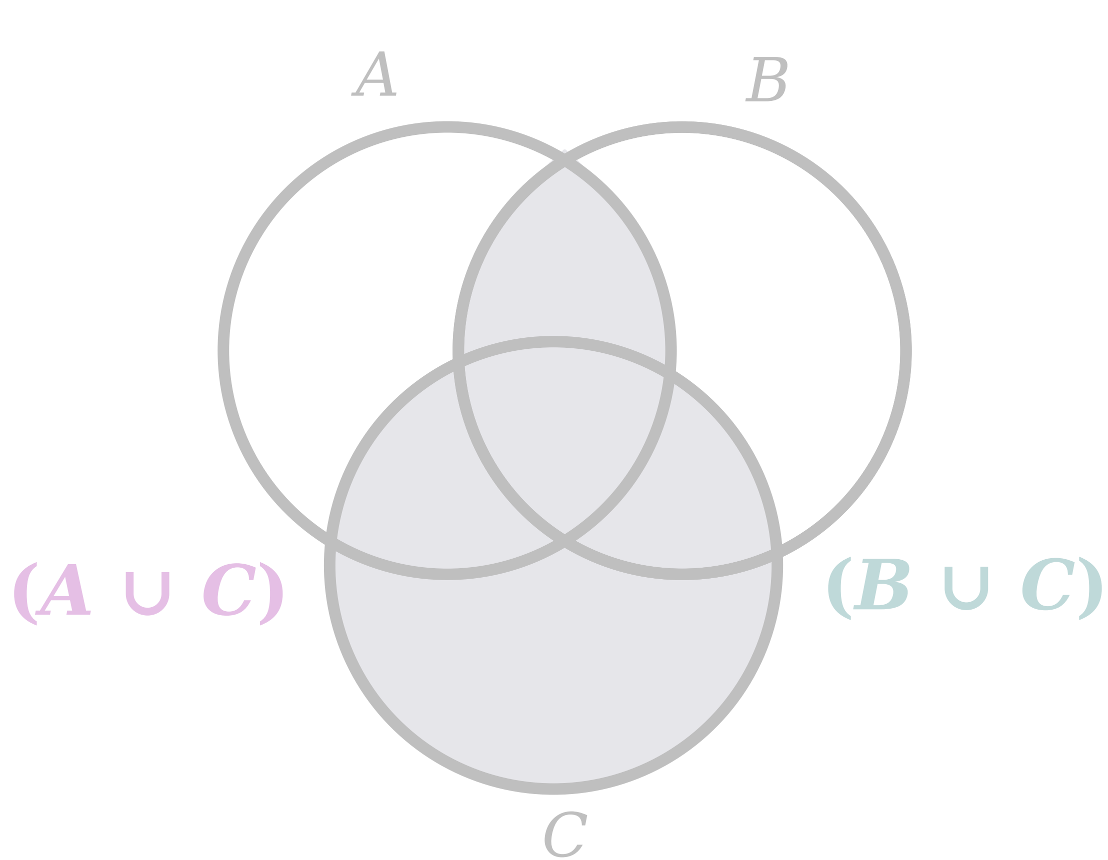

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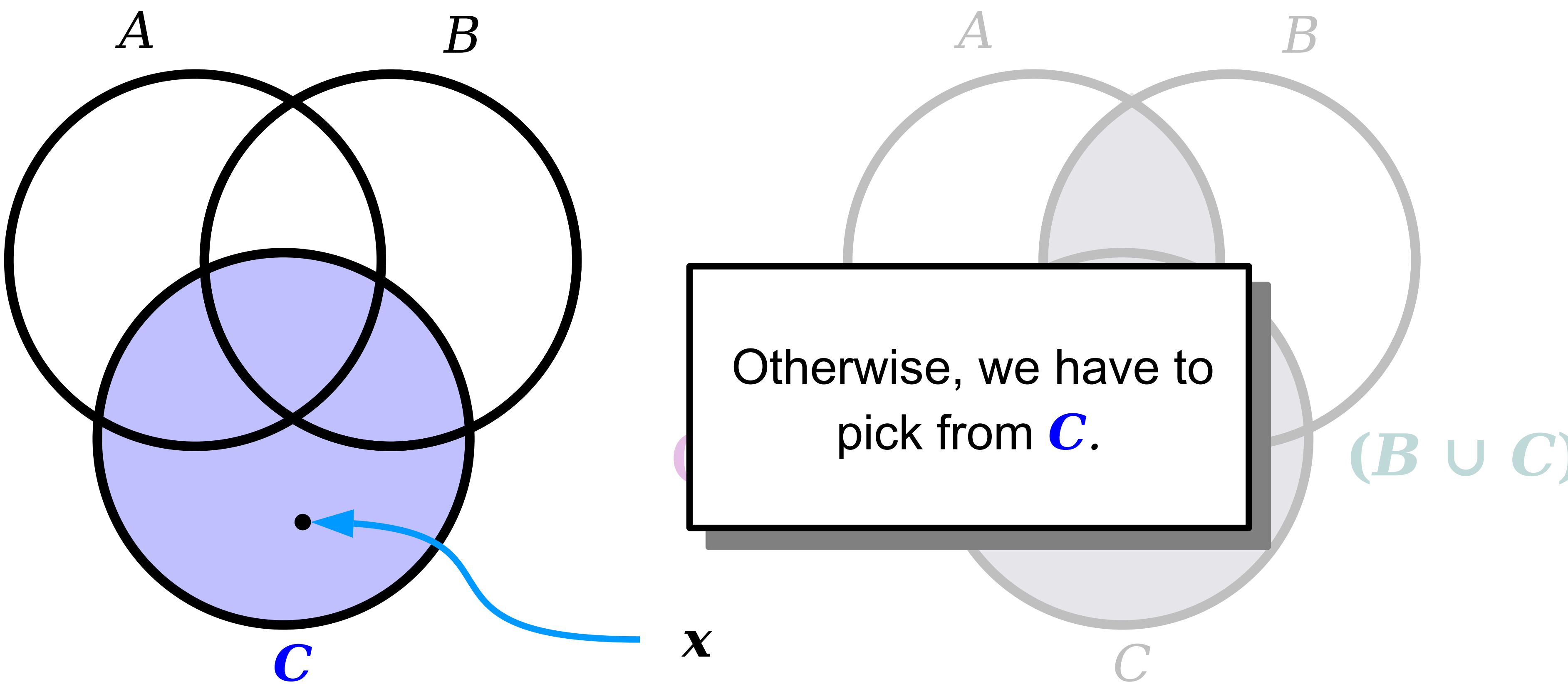

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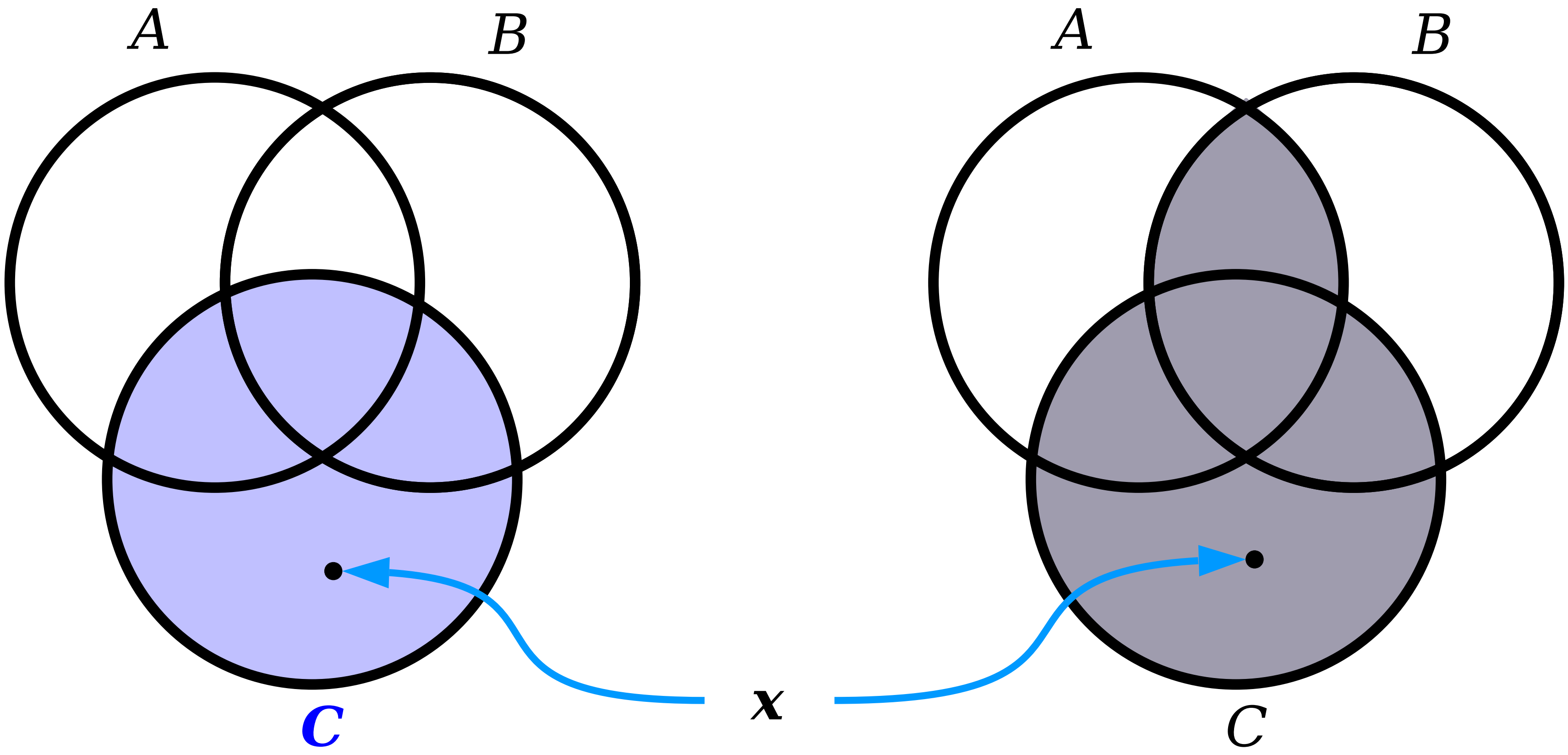

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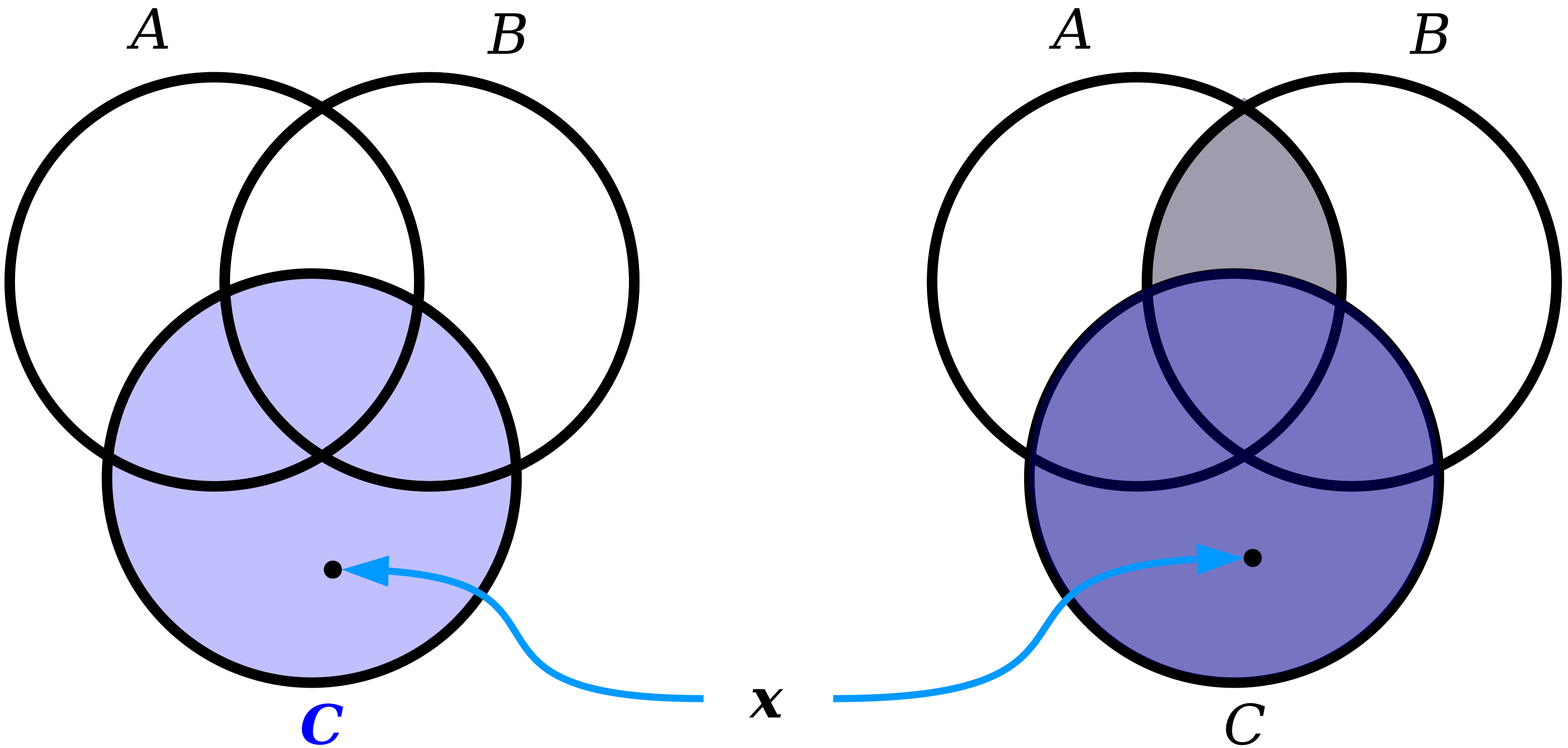

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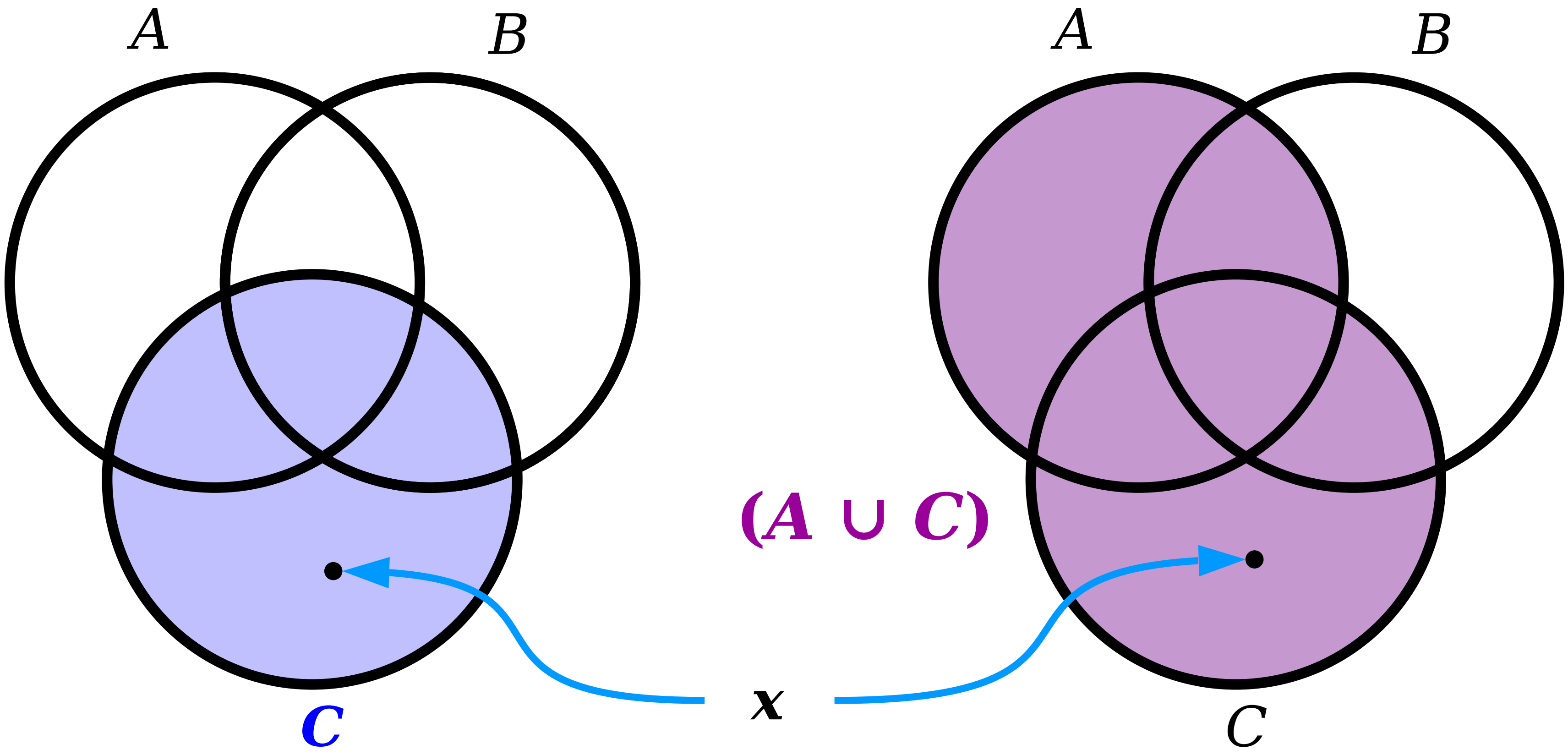

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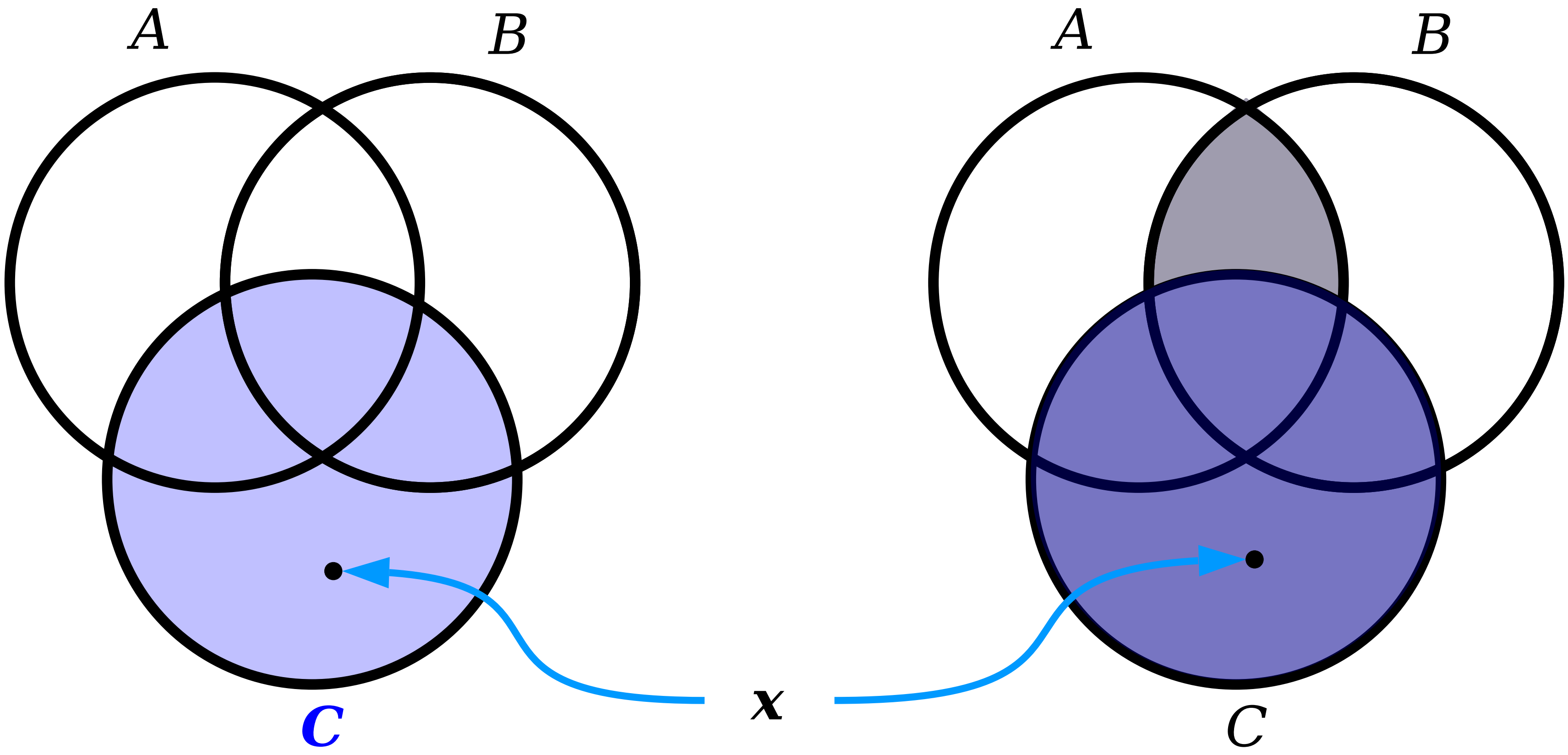

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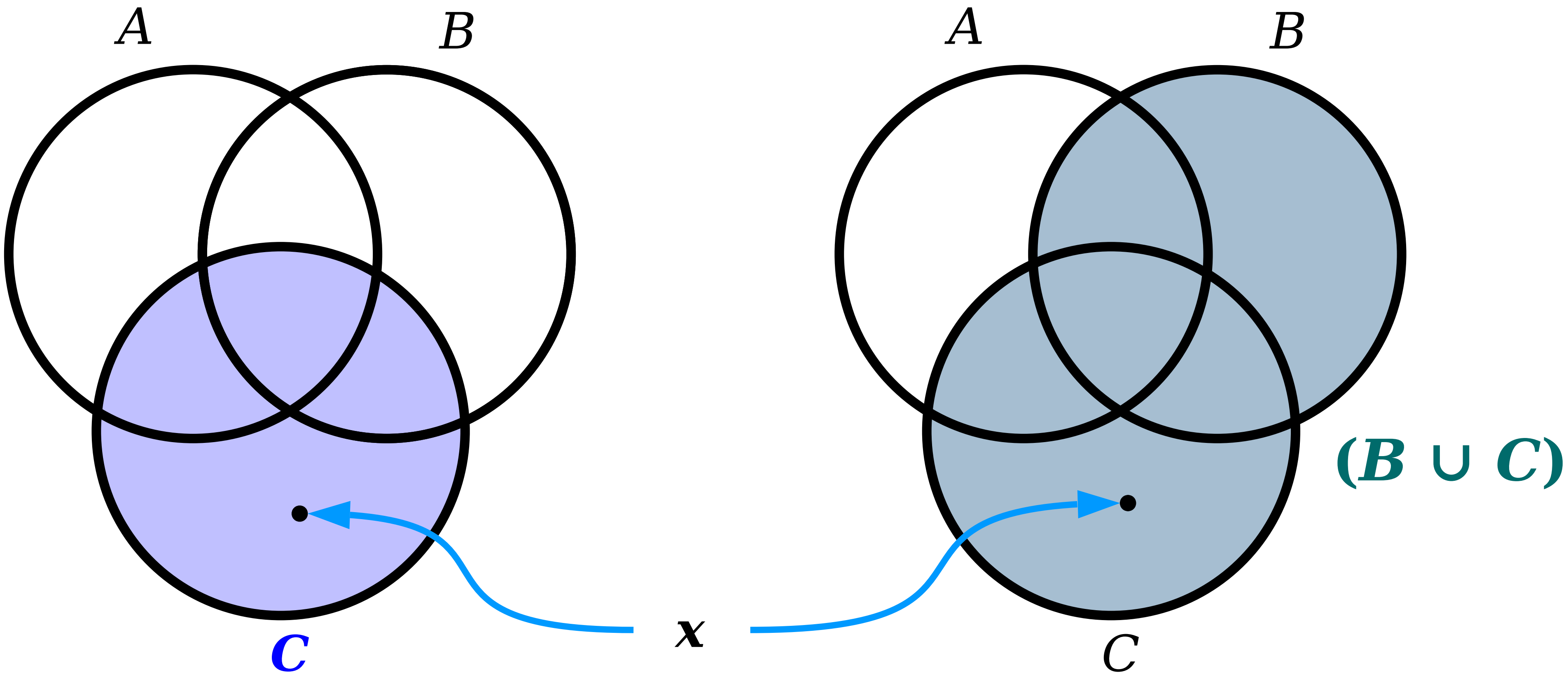

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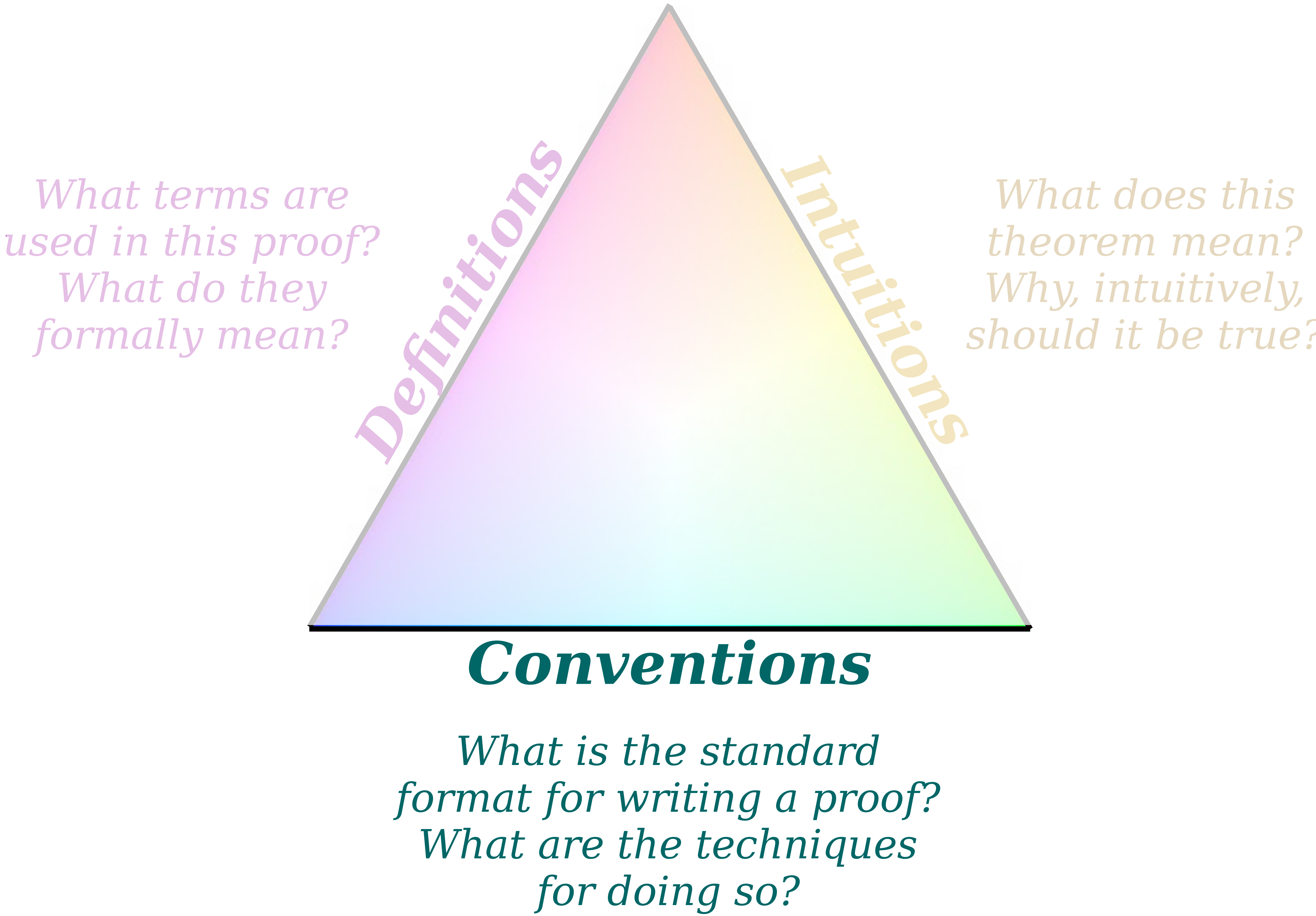
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Proof:

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Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$.

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Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

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Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

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Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$.

Case 2: $x \in A \cap B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since x … at $x \in C$.
We cons…

Case … and
tha…

Case … at
x …
$x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

These are ***arbitrary choices***. Rather than specifying what A, B, C, and x are, we're signaling to the reader that they could, in principle, supply any choices of A, B, C, and x that they'd like.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $\boldsymbol{x \in C}$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $\boldsymbol{x \in A \cap B}$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that

In eit

estab

If you know that $\boldsymbol{x \in S \cup T}$***:***
You can conclude that $x \in S$ or that $x \in T$ (or both).

If you know that $\boldsymbol{x \in S \cap T}$***:***
You can conclude both that $x \in S$ and that $x \in T$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$,

To prove that $x \in S \cup T$:
Prove either that $x \in S$ or that $x \in T$ (or both).

To prove that $x \in S \cap T$:
Prove both that $x \in S$ and that $x \in T$.

Case 1: $\boldsymbol{x \in C}$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $\boldsymbol{x \in A \cap B}$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that x …
$x \in A \cup C$ and th…

In either case, we lear…
establishes that $x \in (A$…

This is called a **proof by cases** (alternatively, a **proof by exhaustion**) and works by showing that the theorem is true regardless of what specific outcome arises.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary[…]y $x \in (A \cap B) \cup C$. We w[…]

Since $x \in (A \cap B) \cup C$, […]$\in C$. We consider each case[…]

After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

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In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Proofs as a Dialog

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

Proofs as a Dialog

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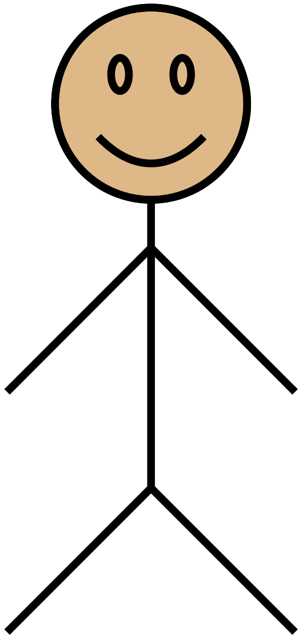

Proof Writer (You)

Proofs as a Dialog

Let n be an arbitrary odd integer.

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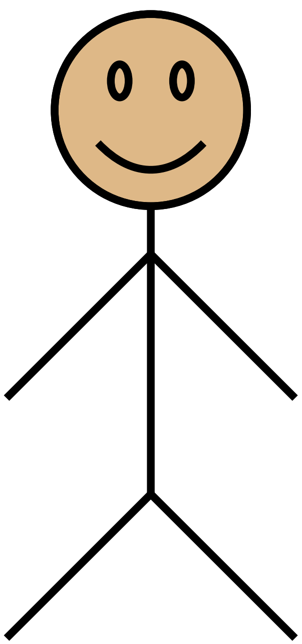
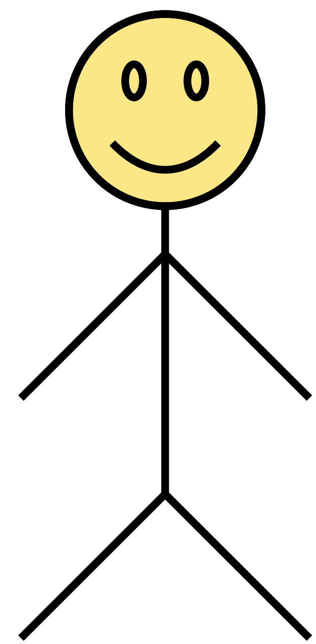


Proof Writer (You)



Proof Reader

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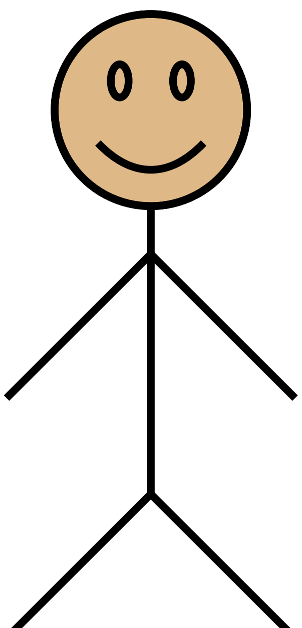
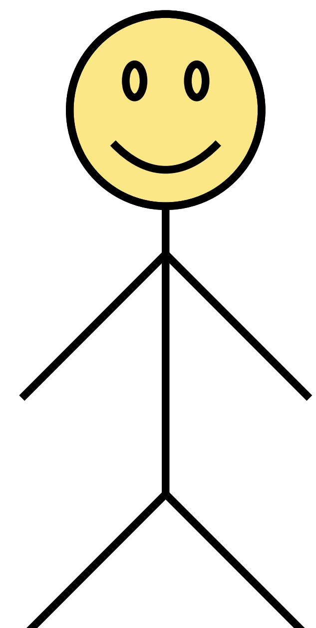


Proof Writer (You)



Proof Reader

Proofs as a Dialog

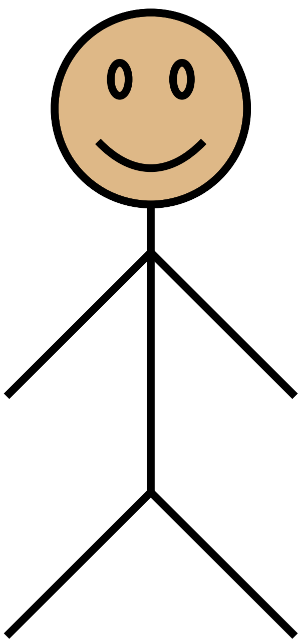
Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

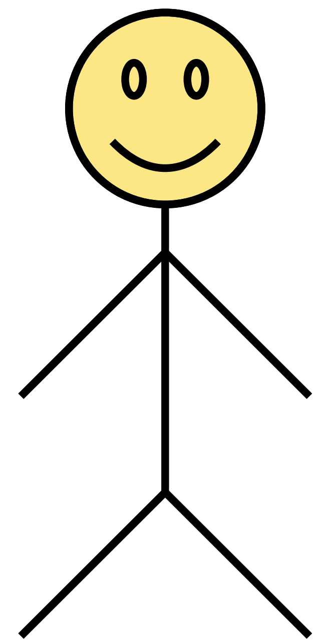
Now, let $z = k - 34$.

$n = 137$

Reader *Picks*



Proof Writer (You)



Proof Reader

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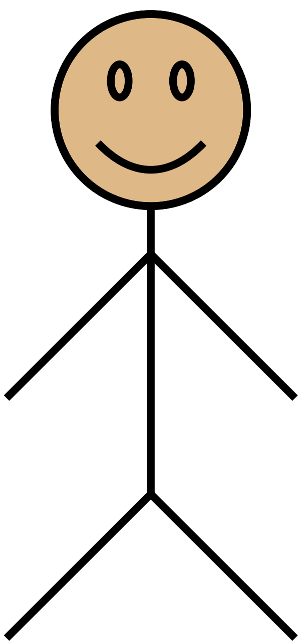
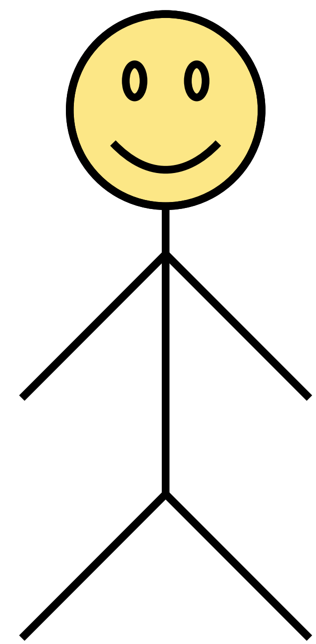
Now, let $z = k - 34$.

$n = 137$

Reader *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

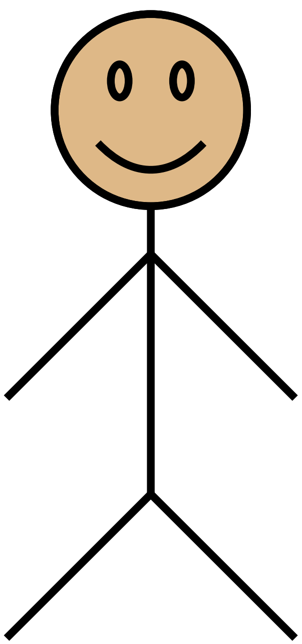
Now, let $z = k - 34$.

$n = 137$

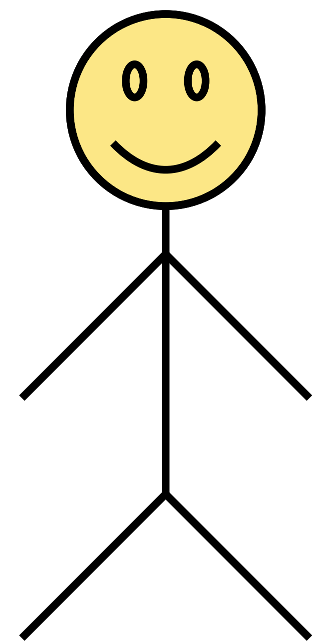
Reader *Picks*

$k = 68$

Neither *Picks*



Proof Writer (You)



Proof Reader

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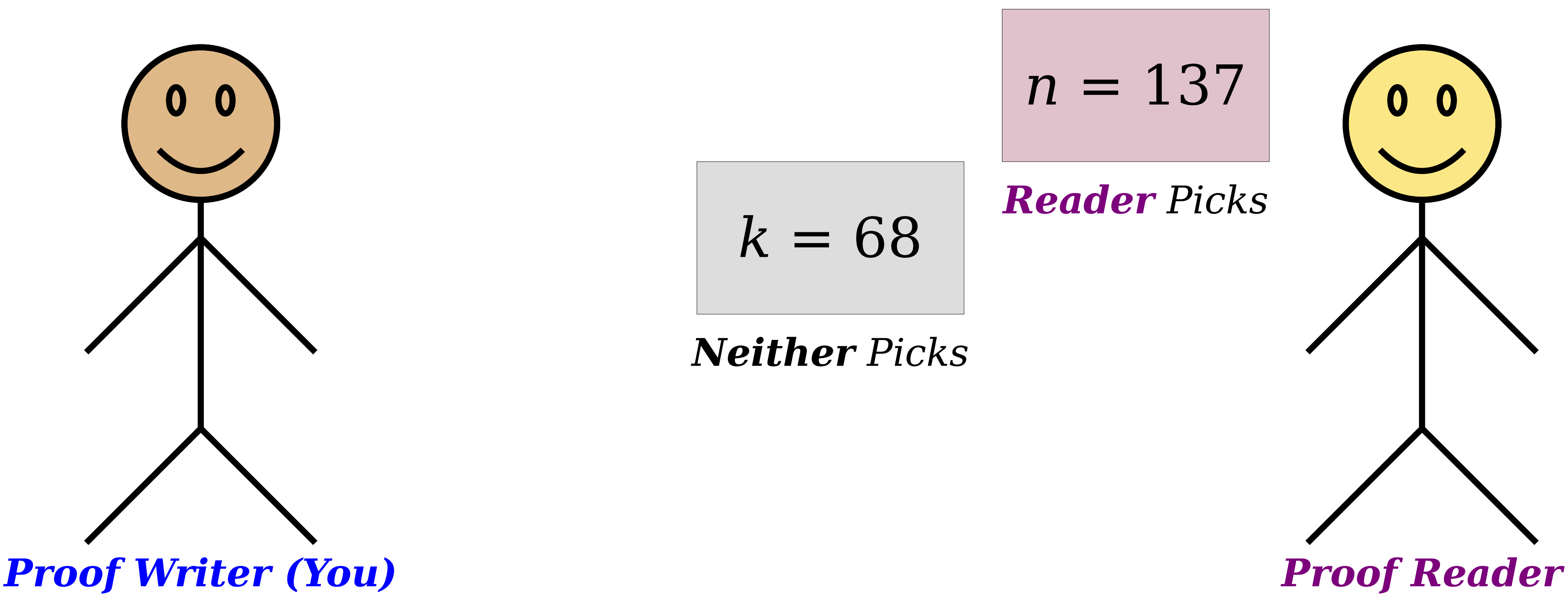

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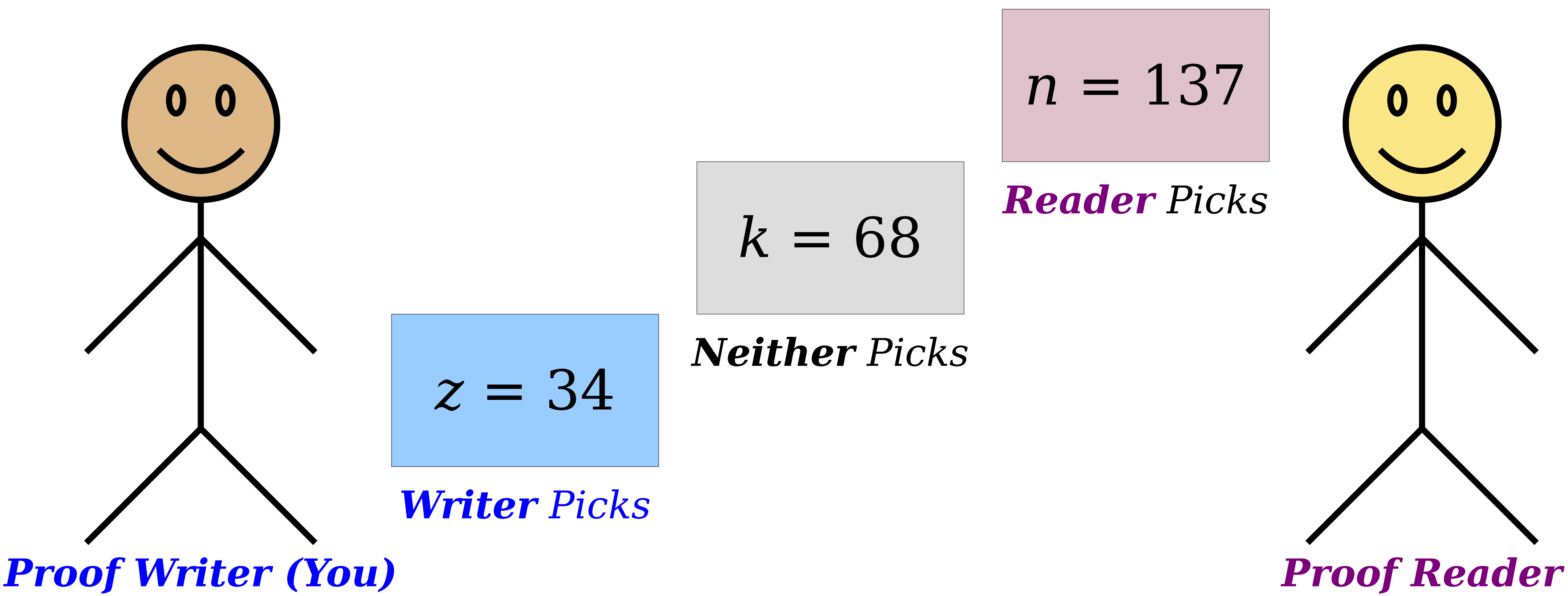

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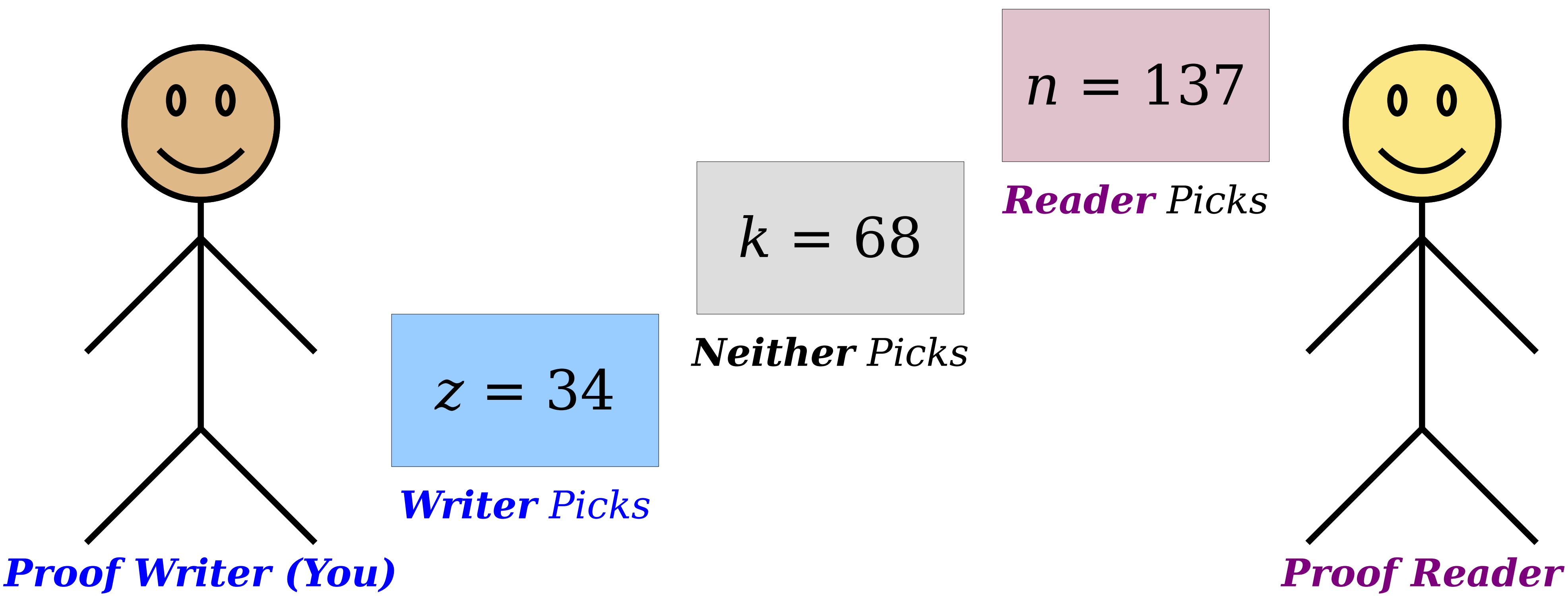

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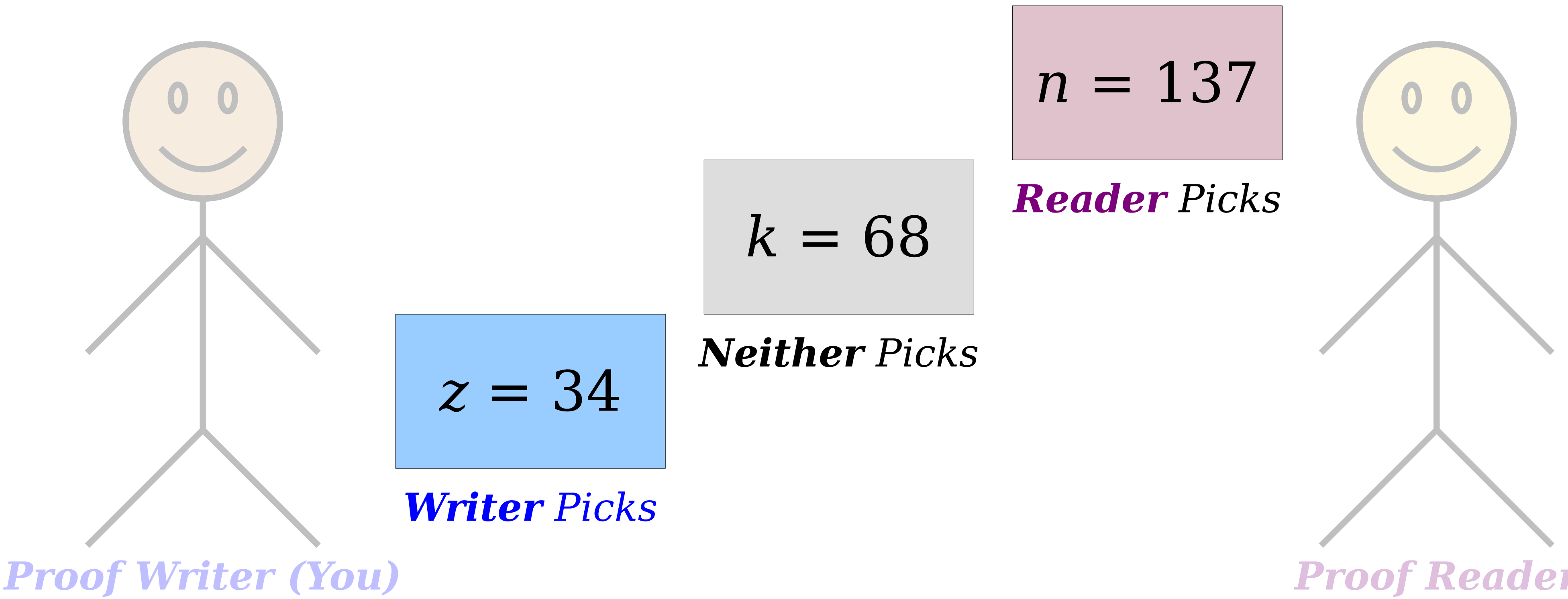
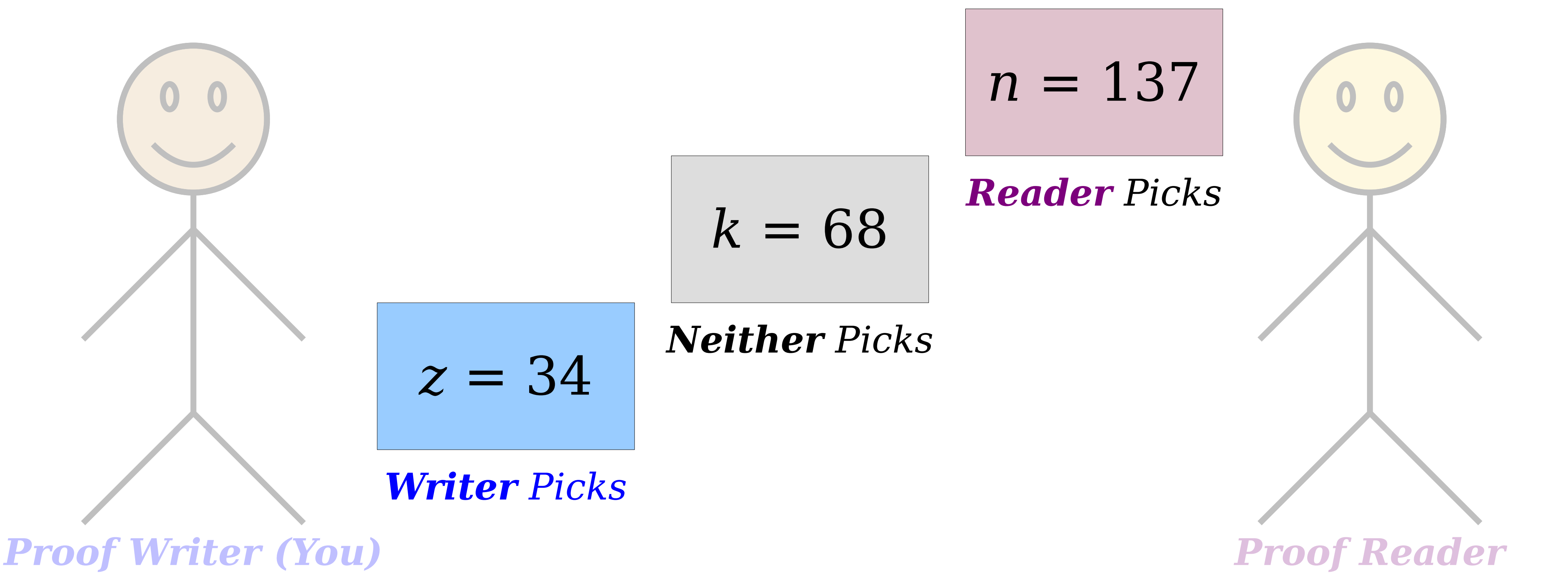


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Sinc… r

Now, let $z = k - 34$.



Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

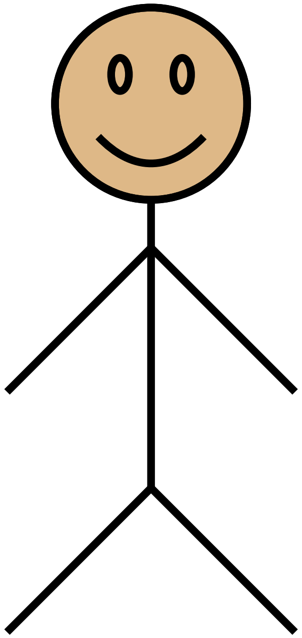
Proofs as a Dialog

⚠

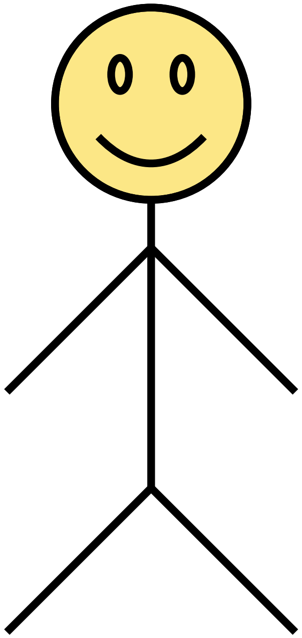
Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.

⚠



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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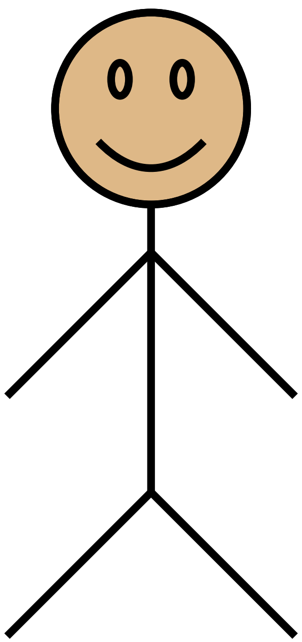
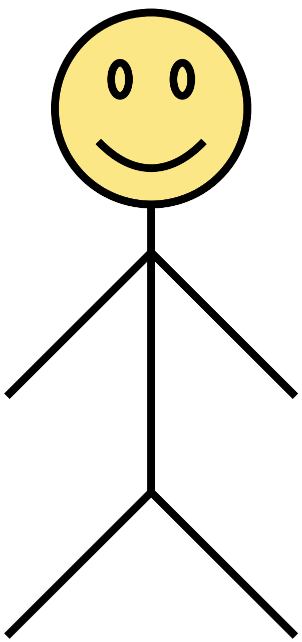
Then for any even x, we know that x+1 is odd.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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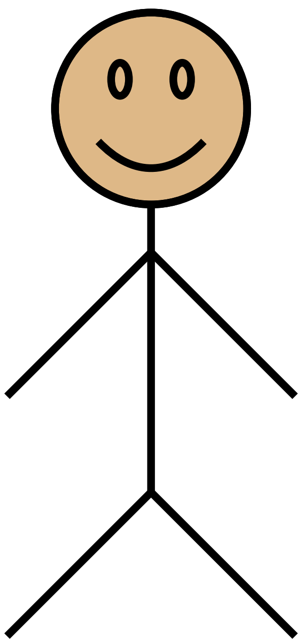
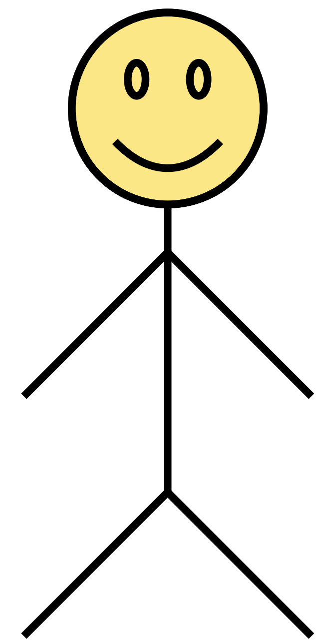


Proof Writer (You)

$x = 242$

Reader *Picks*

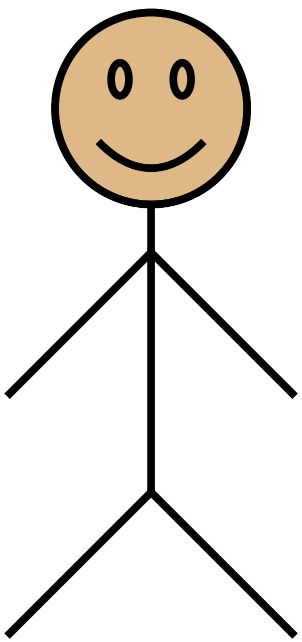


Proof Reader

Proofs as a Dialog

⚠ Let x be an arbitrary even integer. ⚠

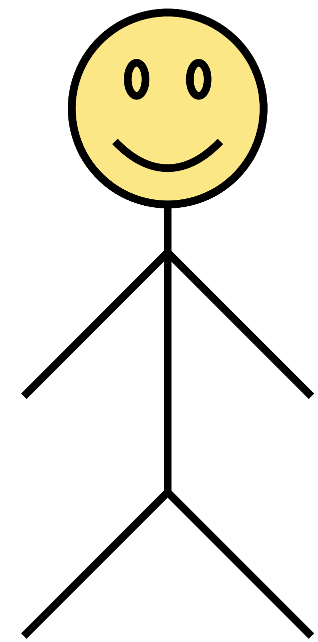
Then for any even x, we know that $x+1$ is odd.



Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

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Let x be an arbitrary even integer.

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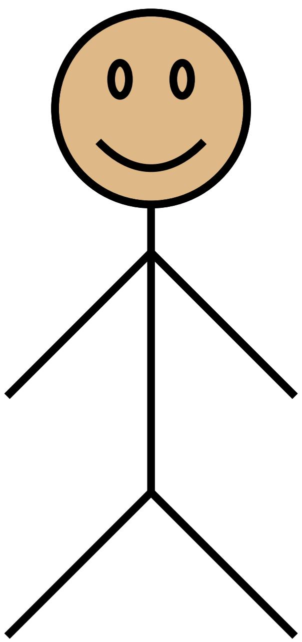
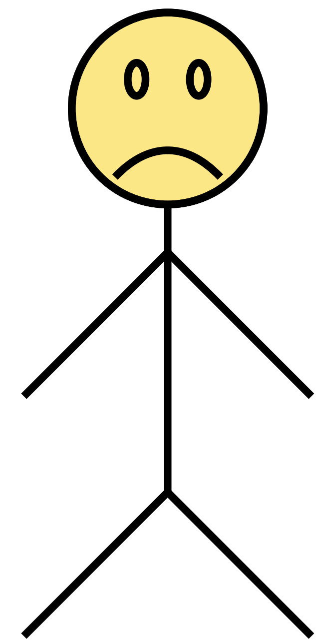


Proof Writer (You)

$x = 242$

Reader *Picks*

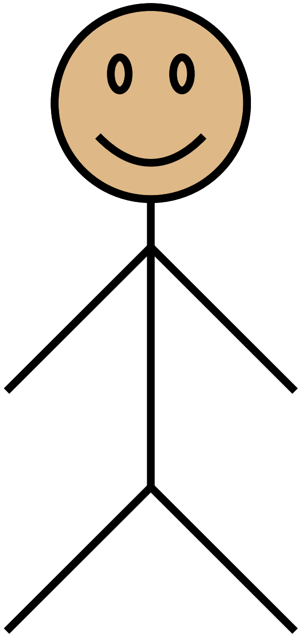


Proof Reader

Proofs as a Dialog

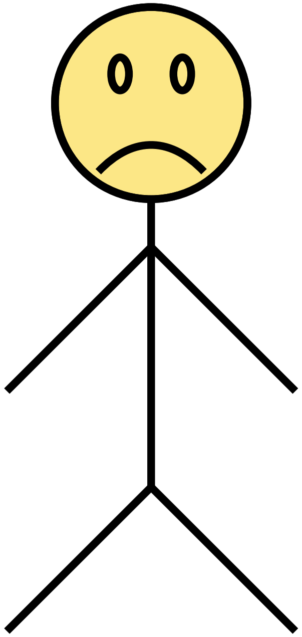
⚠ Let x be an arbitrary even integer. ⚠

Then for any even x, we know that $x+1$ is odd.



$x = 242$

Reader *Picks*



Proof Writer (You)

Proof Reader

Proofs as a Dialog

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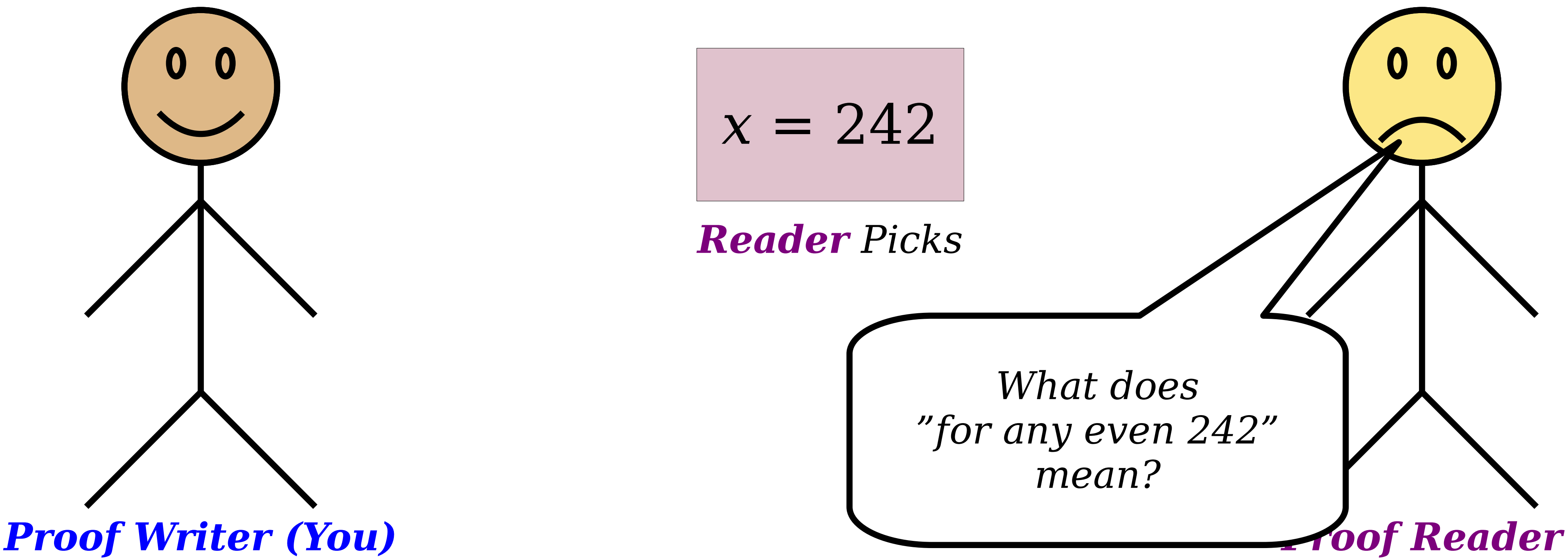

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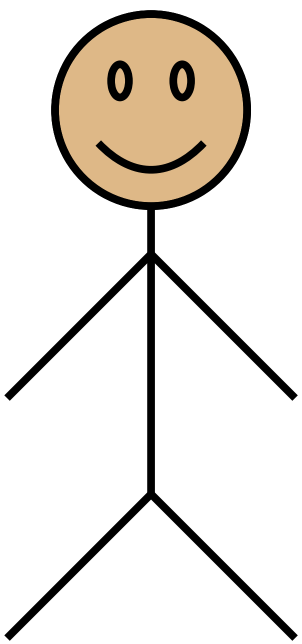
Let x be an arbitrary even integer.

Since x is even, we know that $x+1$ is odd.



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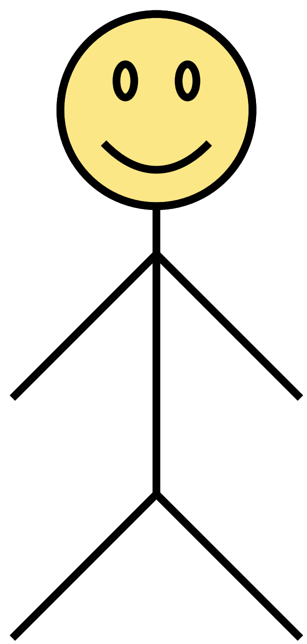

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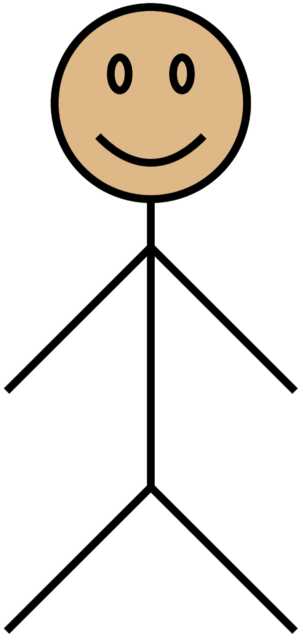

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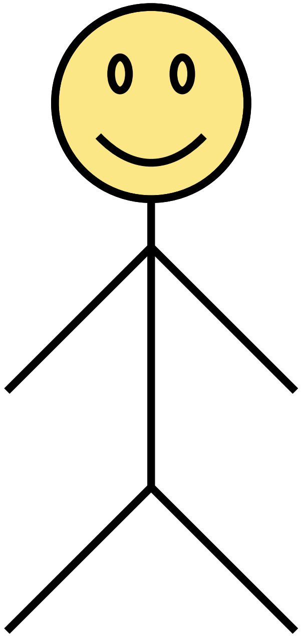

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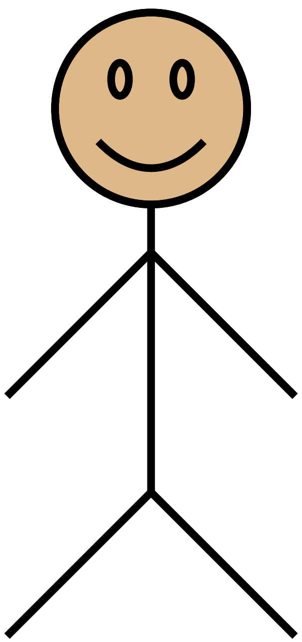

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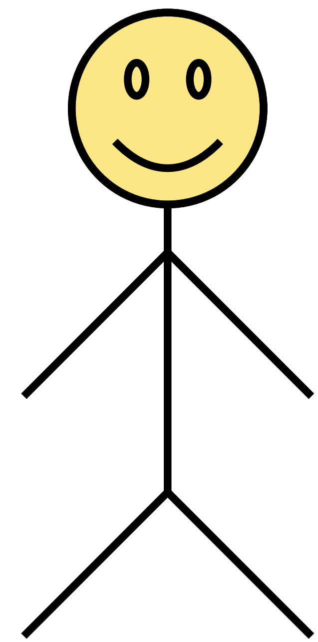

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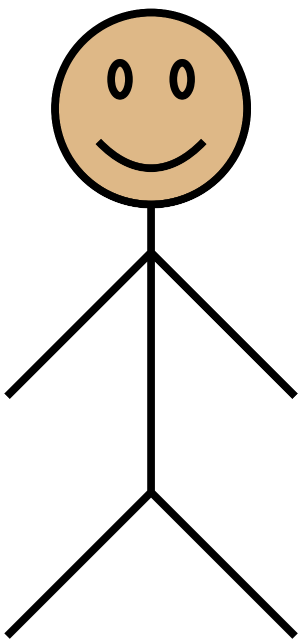

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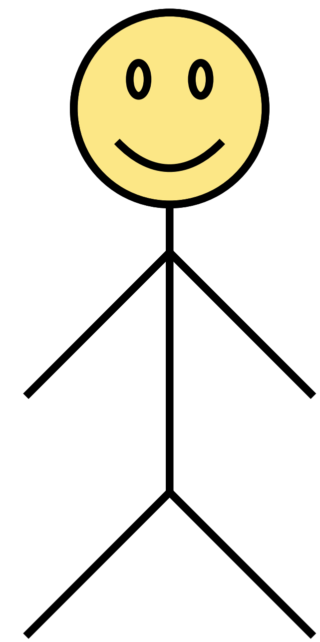


Proof Reader

Every variable needs a value.

Avoid talking about “all x” or “every x” when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

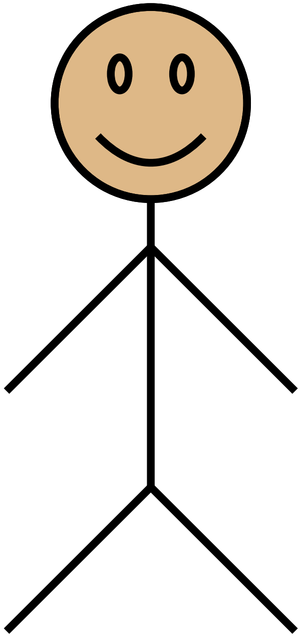
Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog



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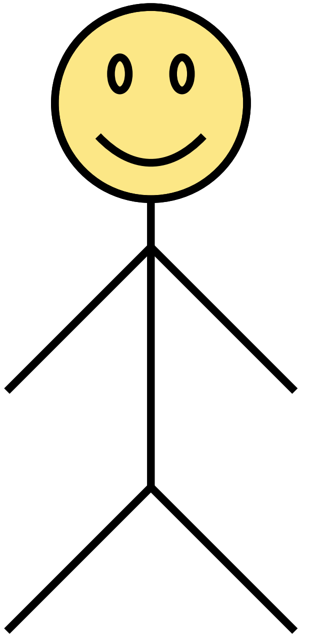


Proof Reader

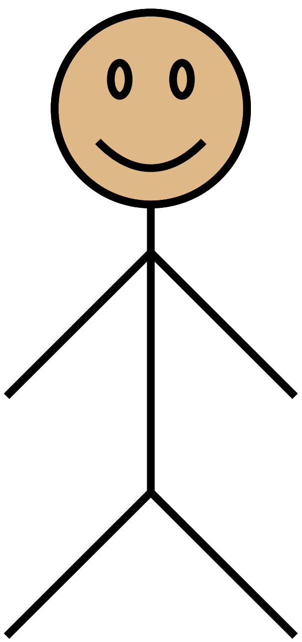
Proofs as a Dialog

⚠

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.

⚠



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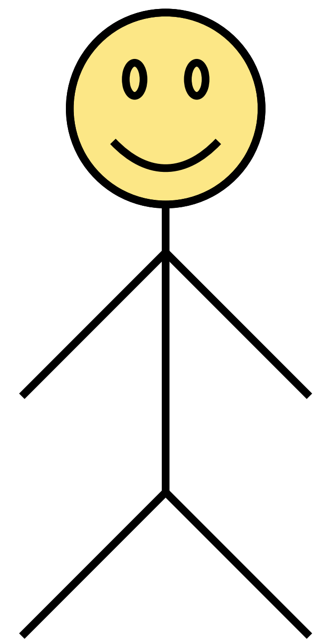

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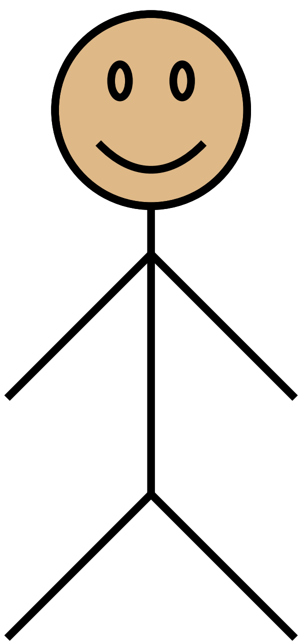

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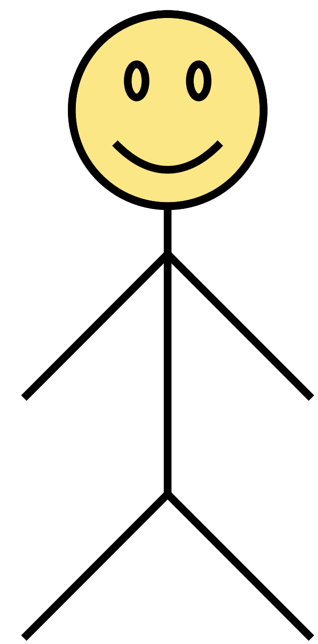

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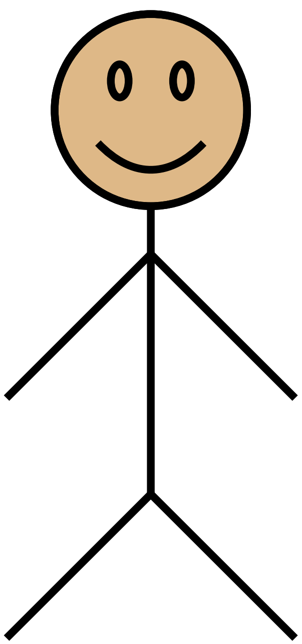


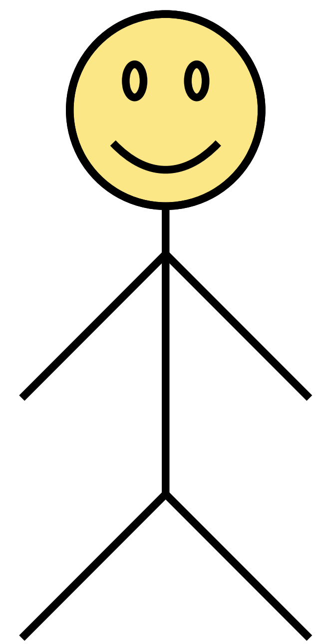
Proof Writer (You)

$m = 103$

Reader *Picks*

$n = 166$

Reader *Picks*



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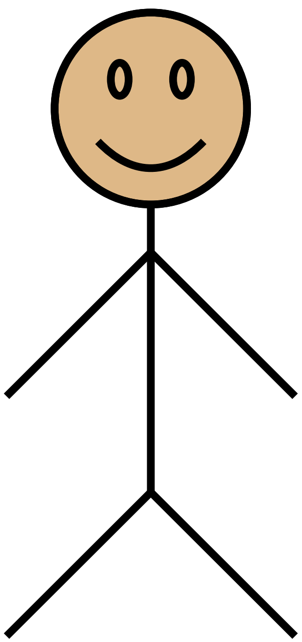

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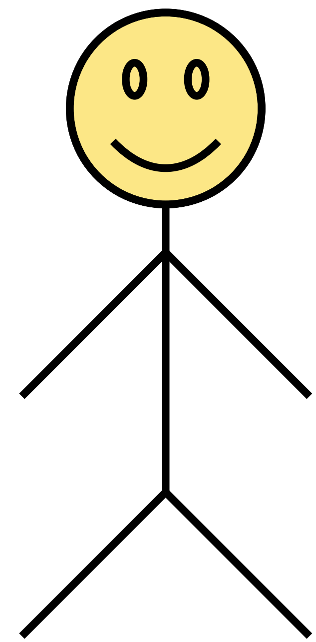

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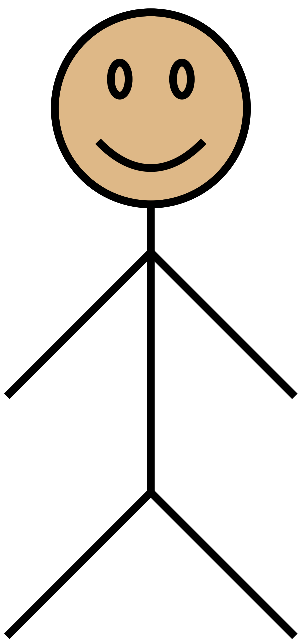

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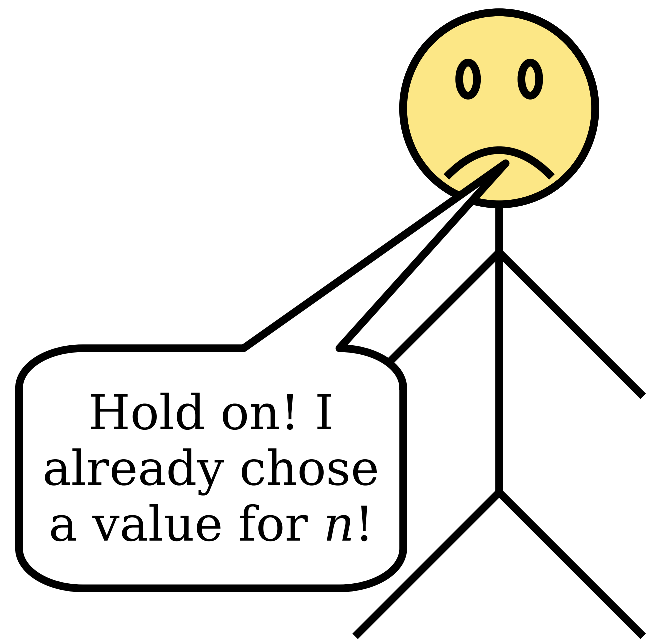


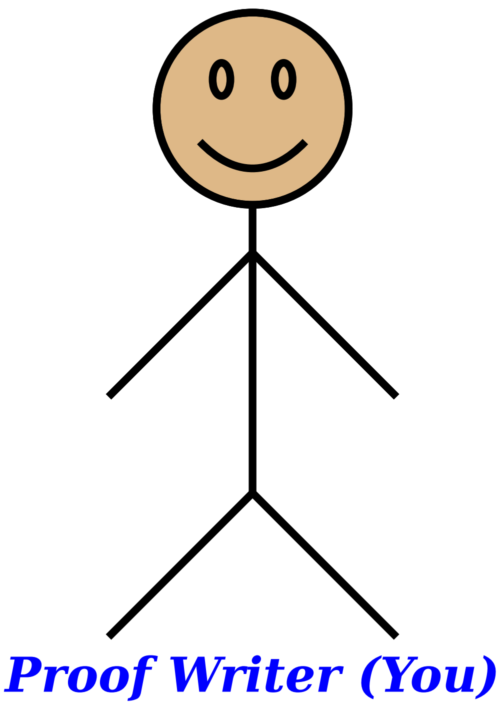
Proof Writer (You)

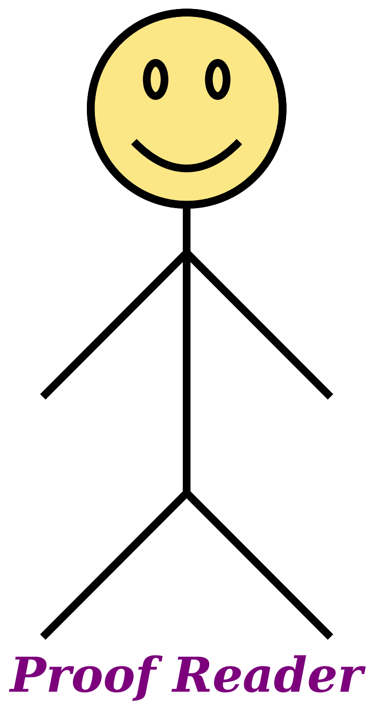
Proof Reader

Proofs as a Dialog

⚠ Let $n = 1$.

Pick any integer m where $m+1$ is odd. ⚠





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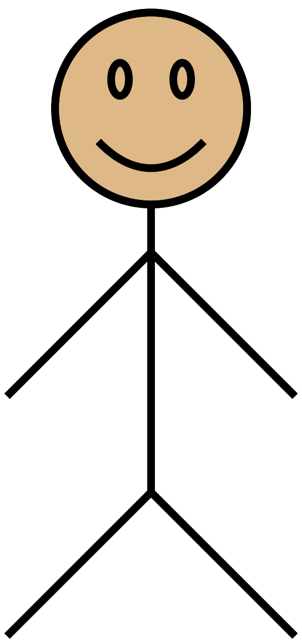

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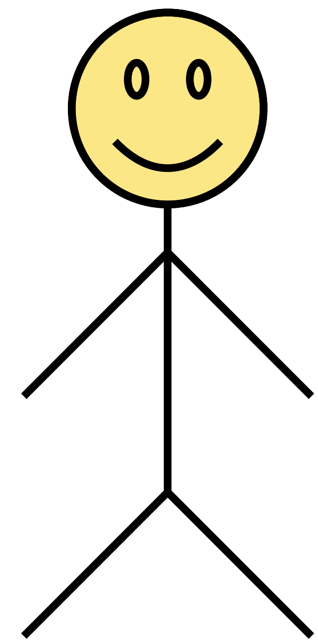

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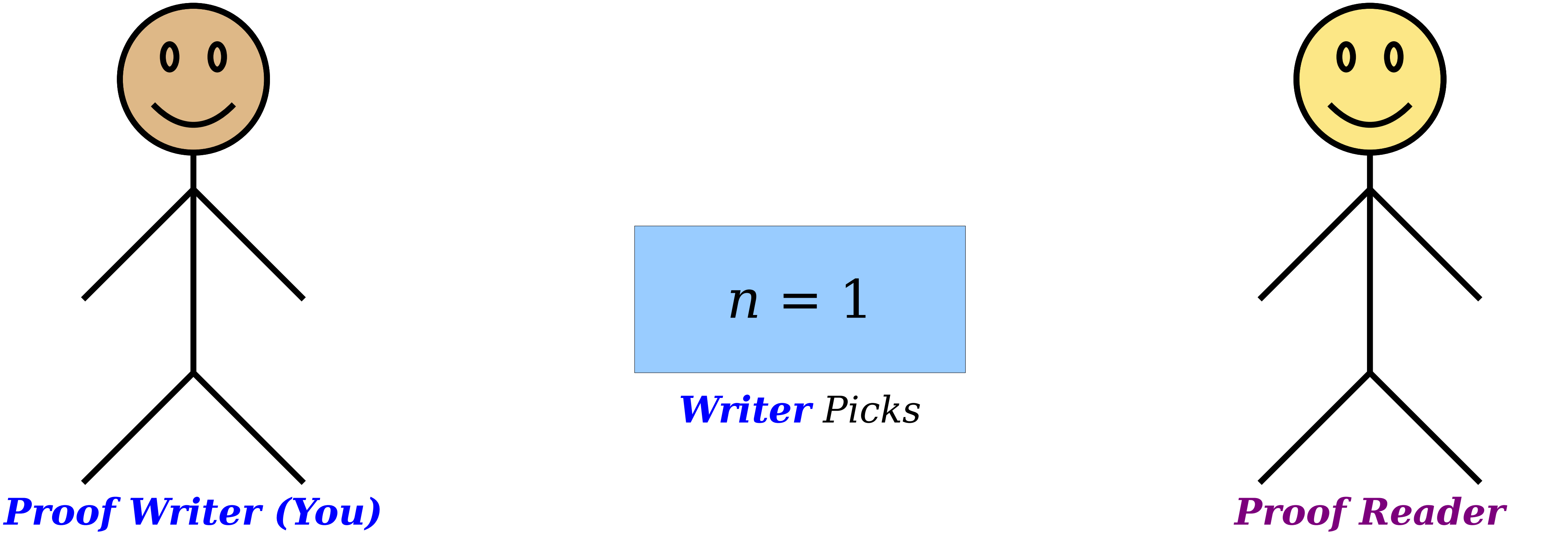

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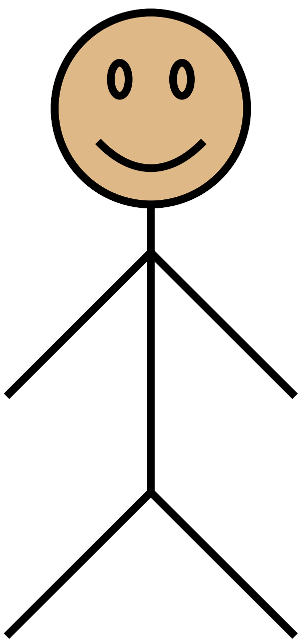
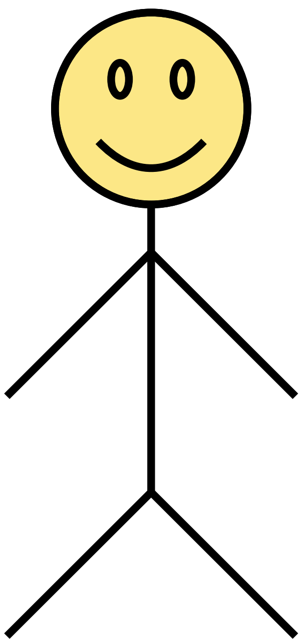


Proof Writer (You)

$n = 1$

Writer *Picks*



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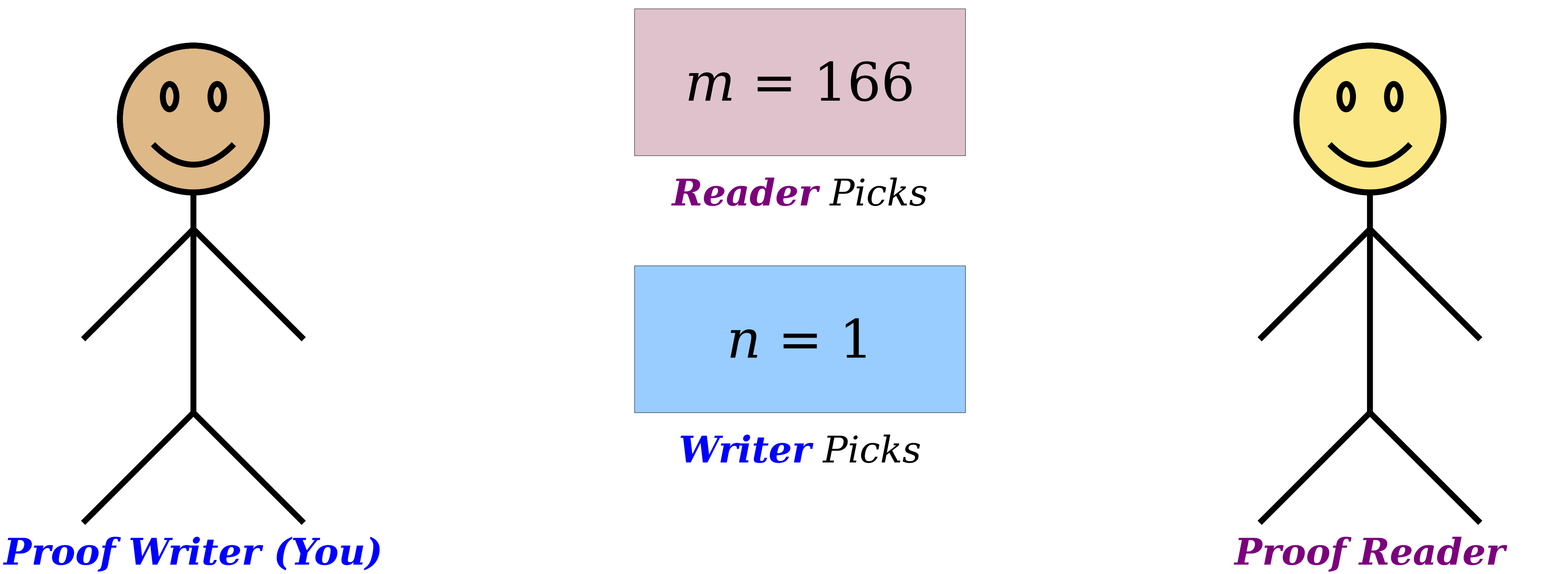

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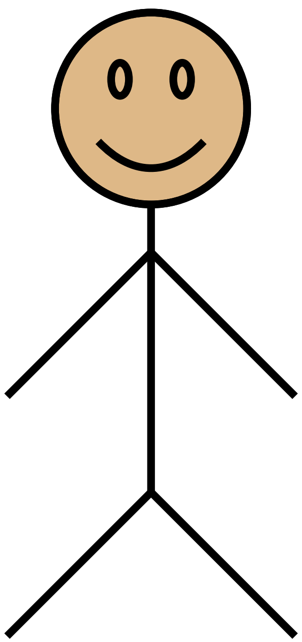


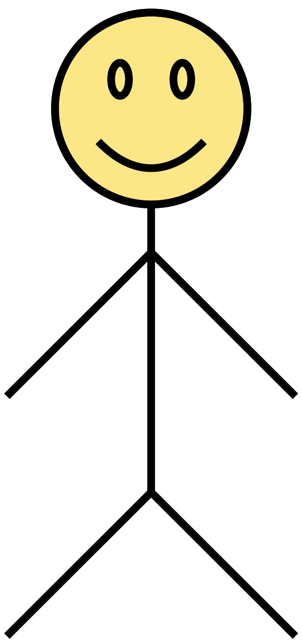
Proof Writer (You)

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



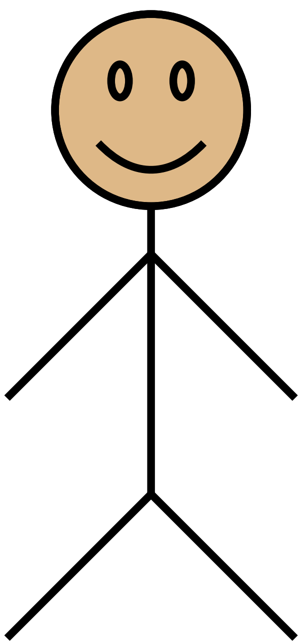
Proof Reader

Proofs as a Dialog

Do we even need n here?

⚠ Let $n = 1$. ⚠

Pick any integer m where $m+1$ is odd.



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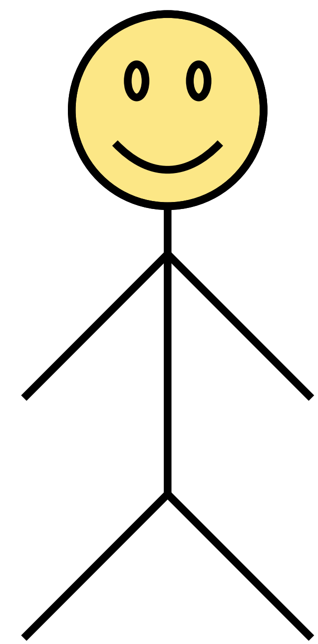

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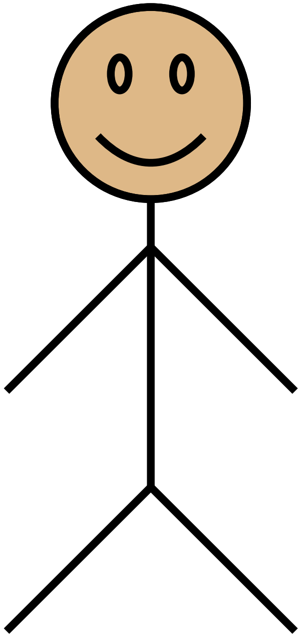

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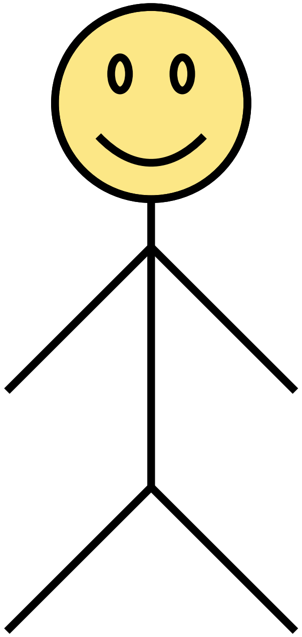

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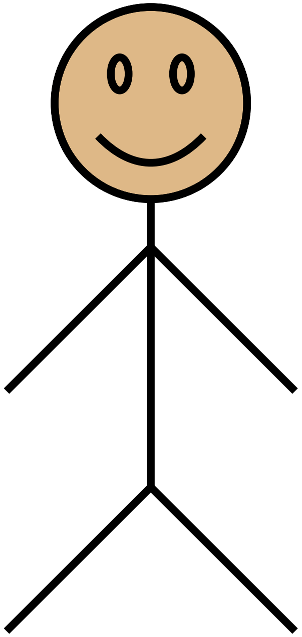

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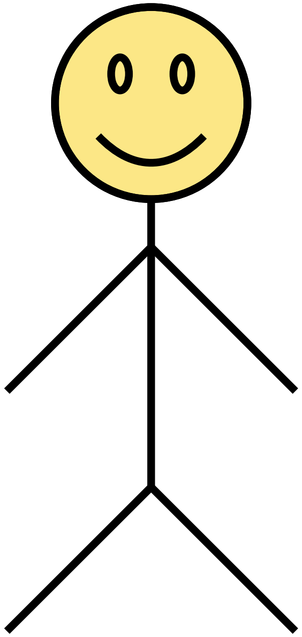


Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.

Your Action Items

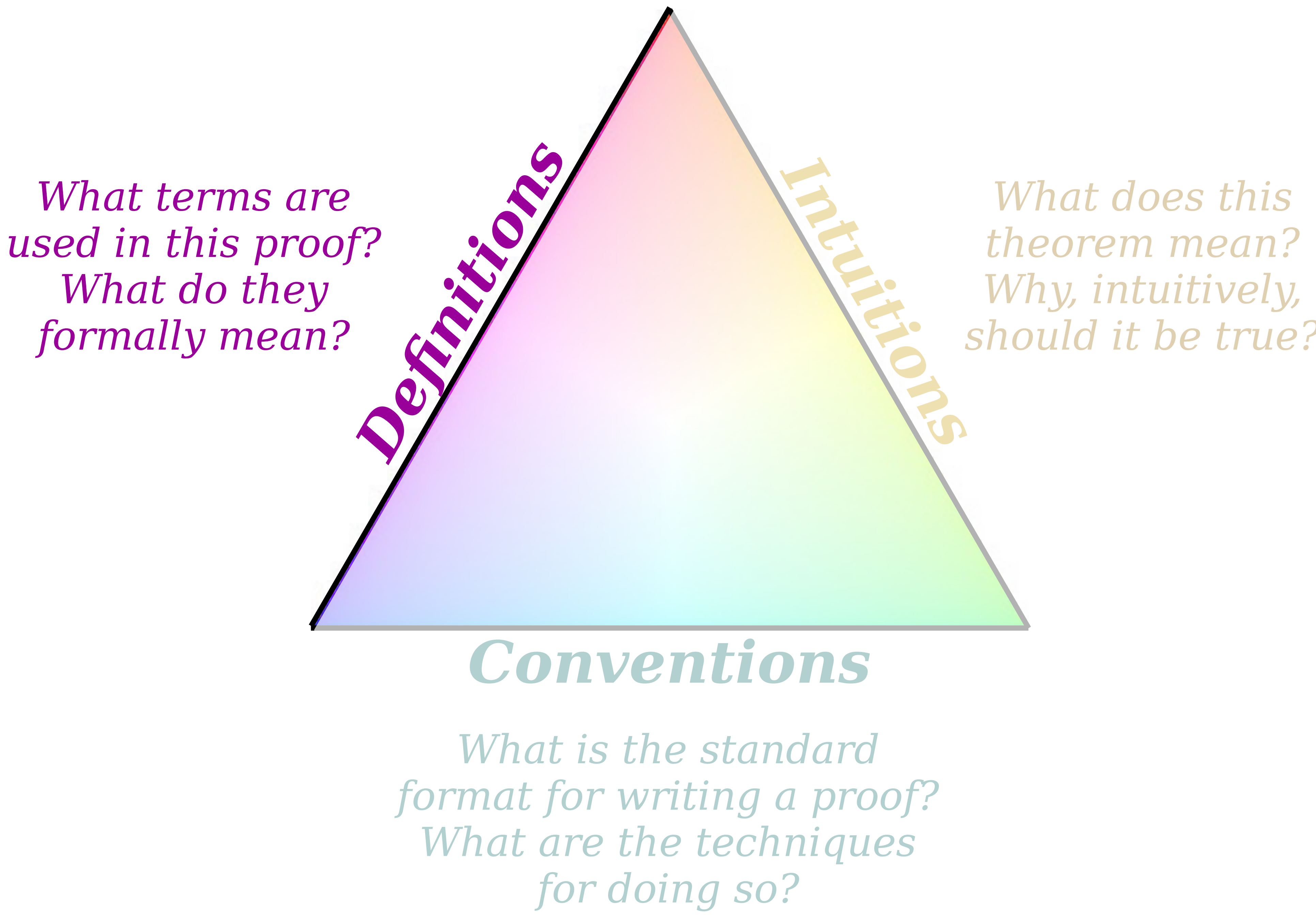
- ***Read "How to Succeed in CS103."***
  - There's a lot of valuable advice in there – take it to heart!
- ***Read "Guide to ∈ and ⊆."***
  - You'll want to have a handle on how these concepts are related, and on how they differ.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Proofs on Subsets

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.


What terms are used in this proof? What do they formally mean?
Definitions
Intuitions
What does this theorem mean? Why, intuitively, should it be true?
Conventions
What is the standard format for writing a proof? What are the techniques for doing so?

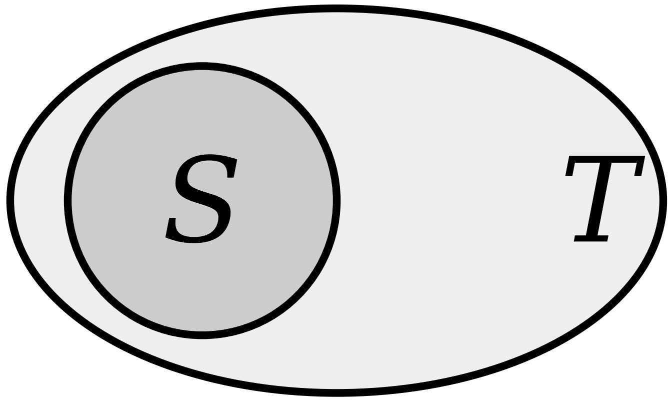
Set Theory Review

- Recall from last time that we write $x \in S$ if x is an element of set S and $x \notin S$ if x is not an element of set S.
- If S and T are sets, we say that S is a subset of T (denoted $S \subseteq T$) if the following statement is true:

For every *x*, if $x \in S$, then $x \in T$.

- What does this mean for proofs?

Subsets



S ⊆ T

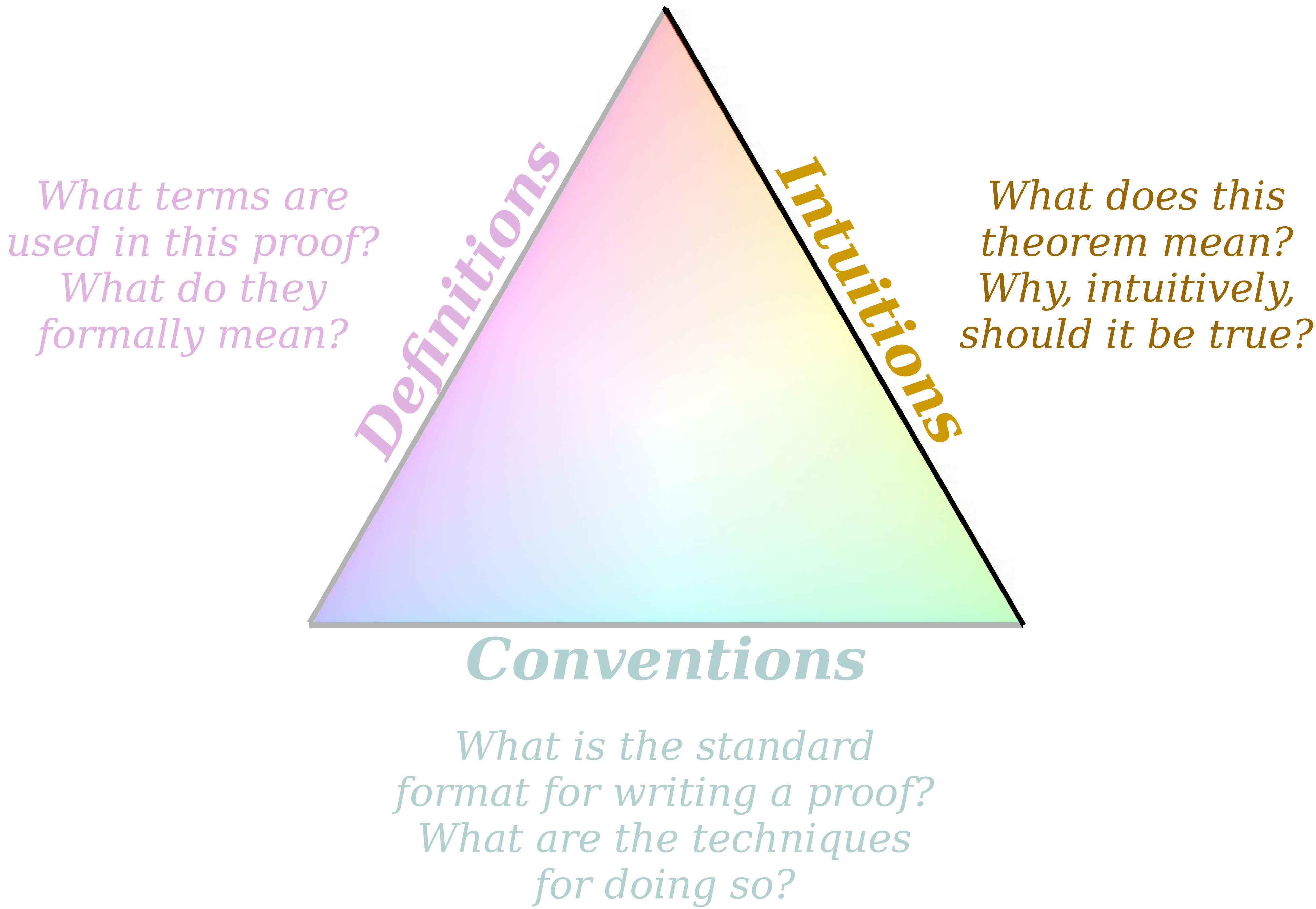
Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that S ⊆ T:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that S ⊆ T:

If you have an $x \in S$, you can conclude $x \in T$.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Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let's Draw Some Pictures!

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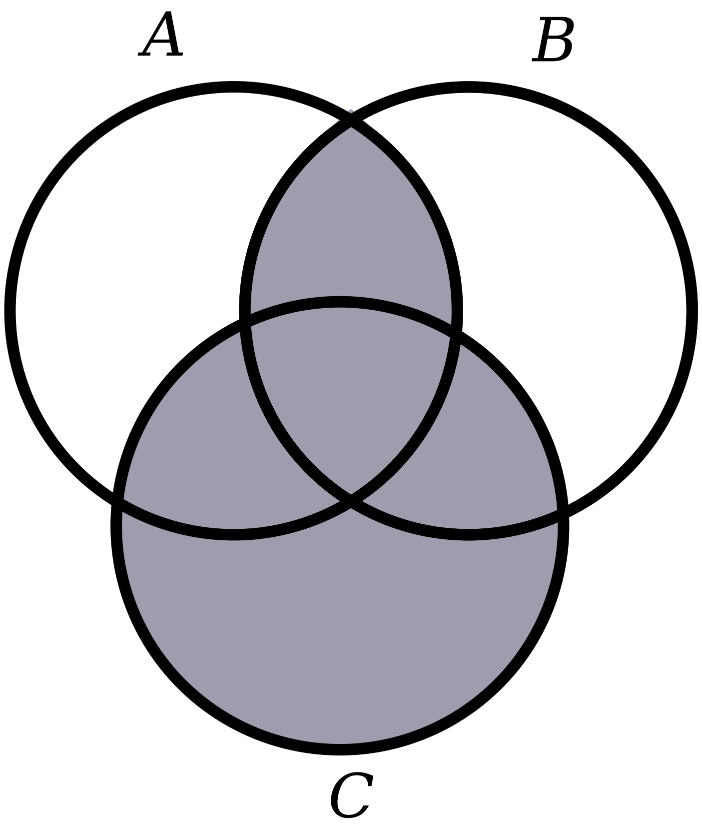

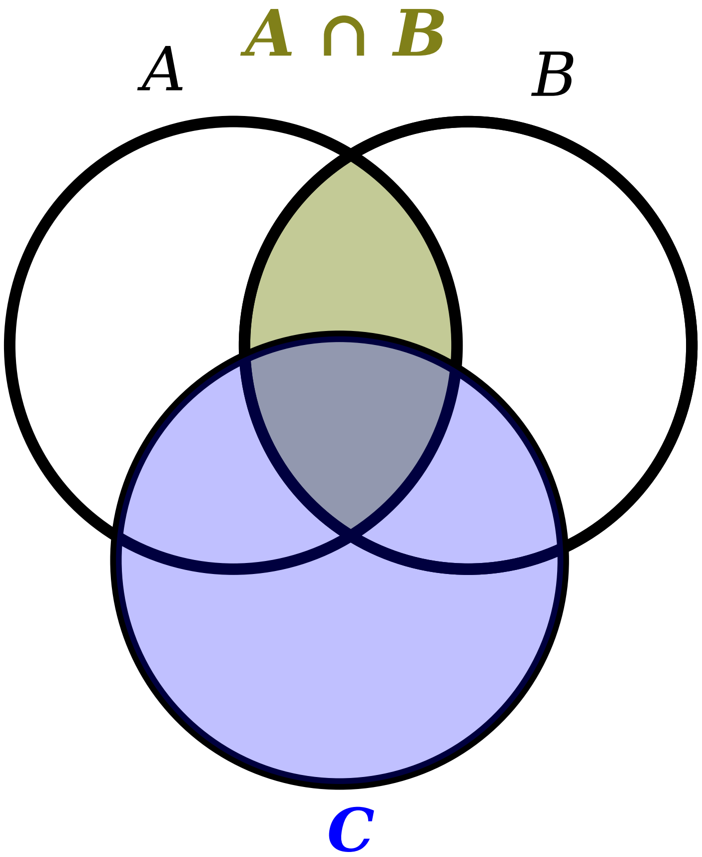

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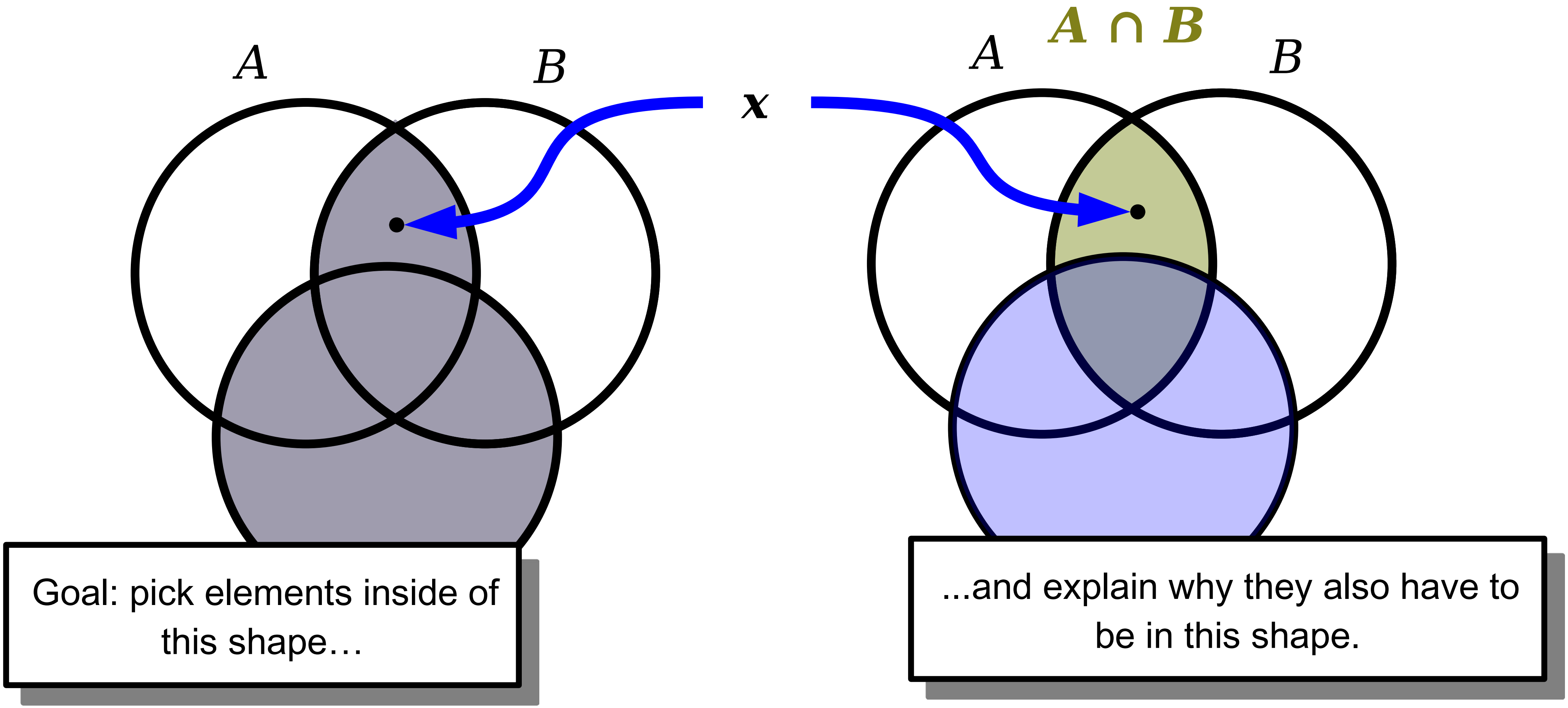

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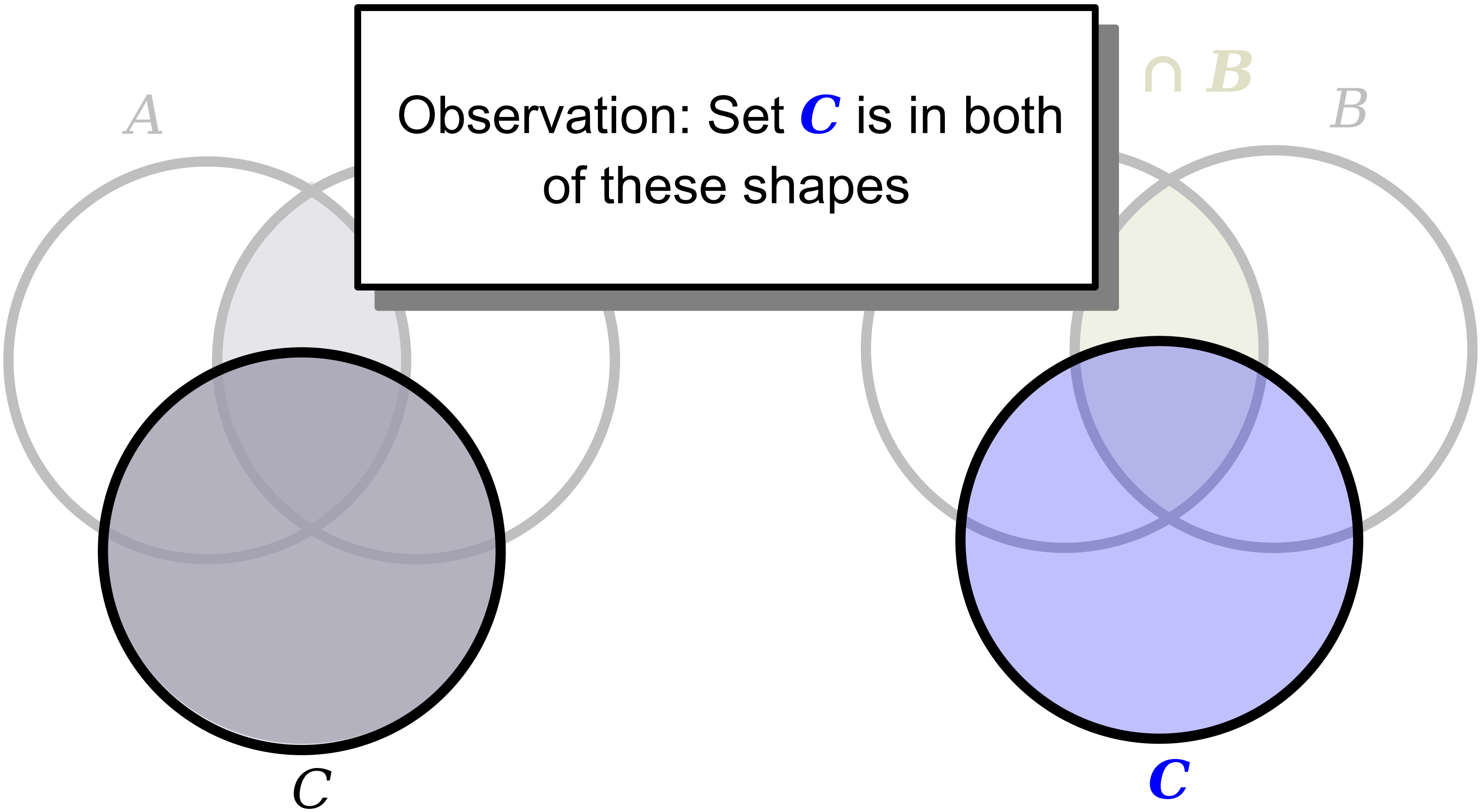

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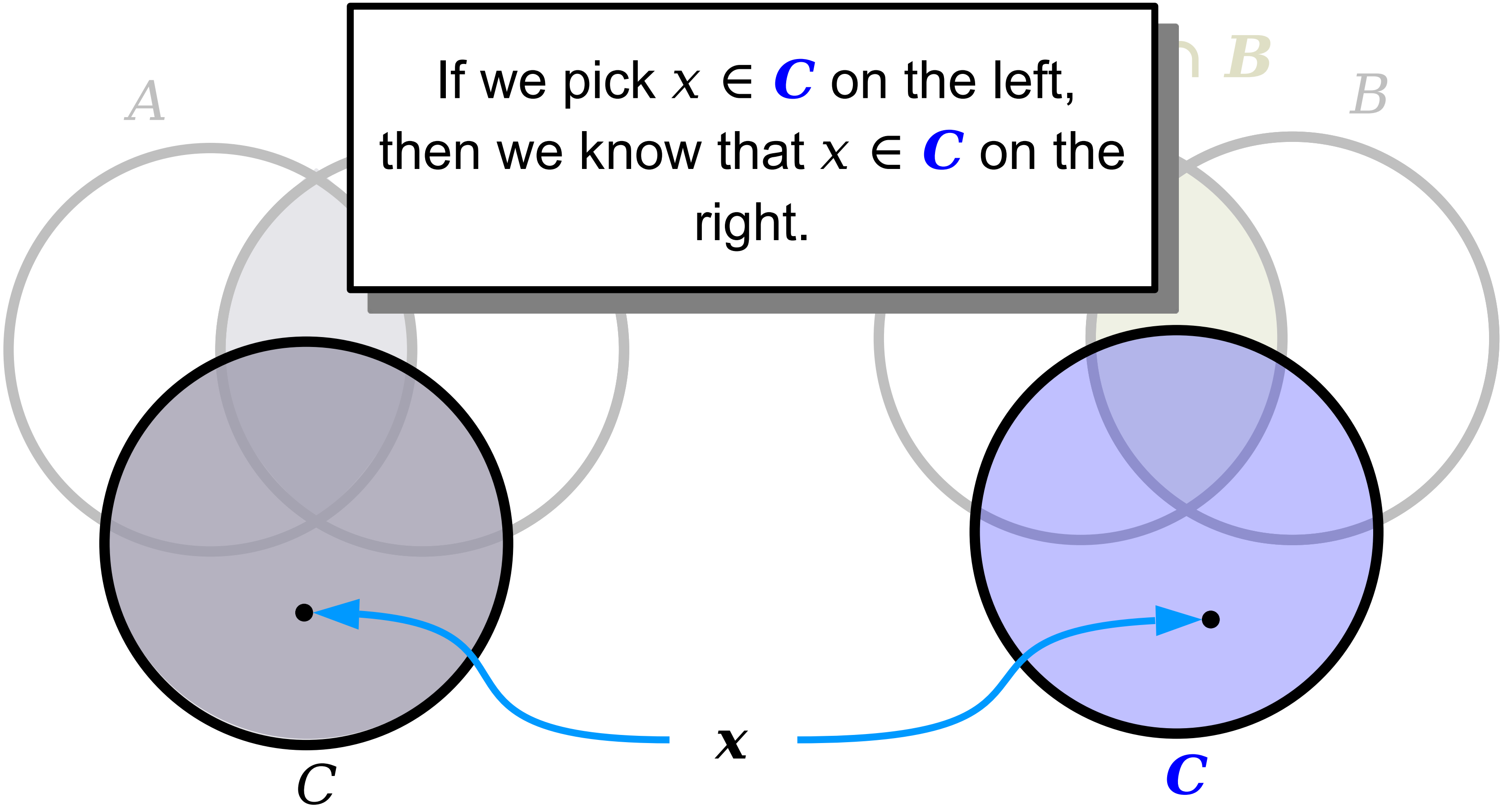

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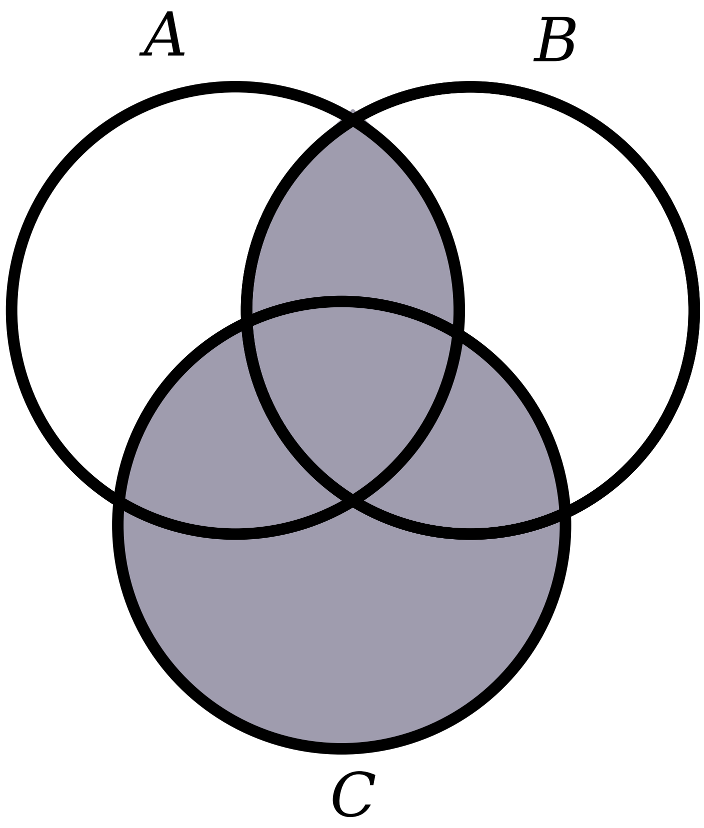

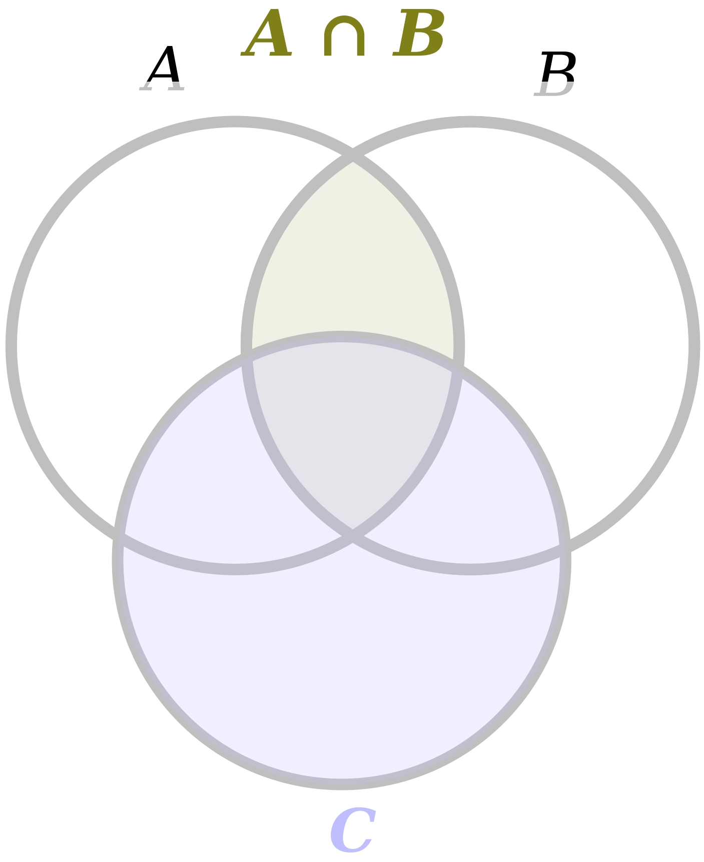

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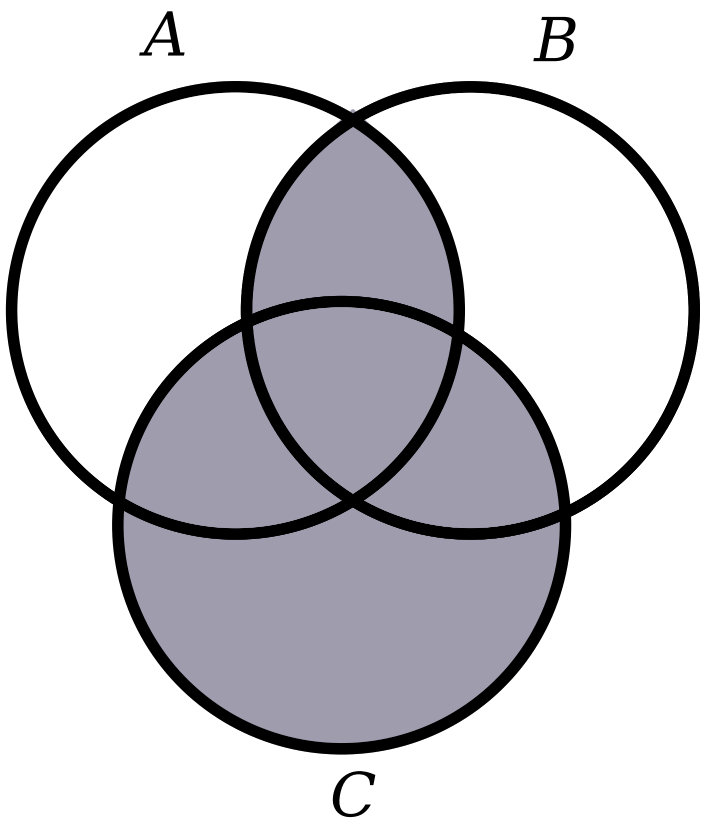

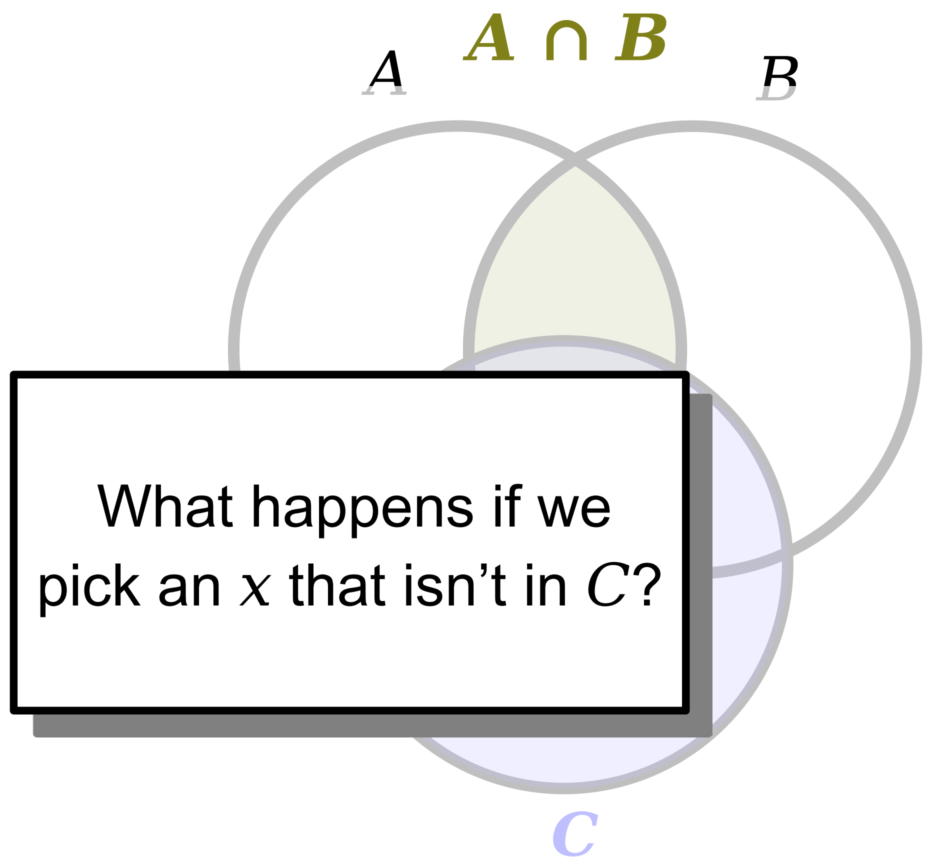

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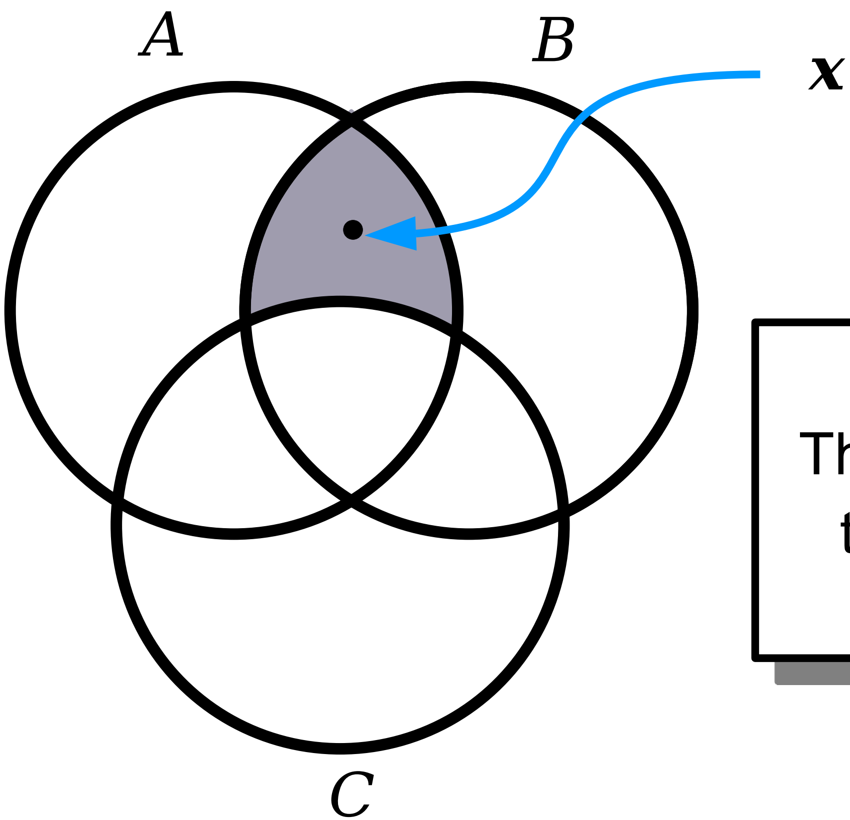
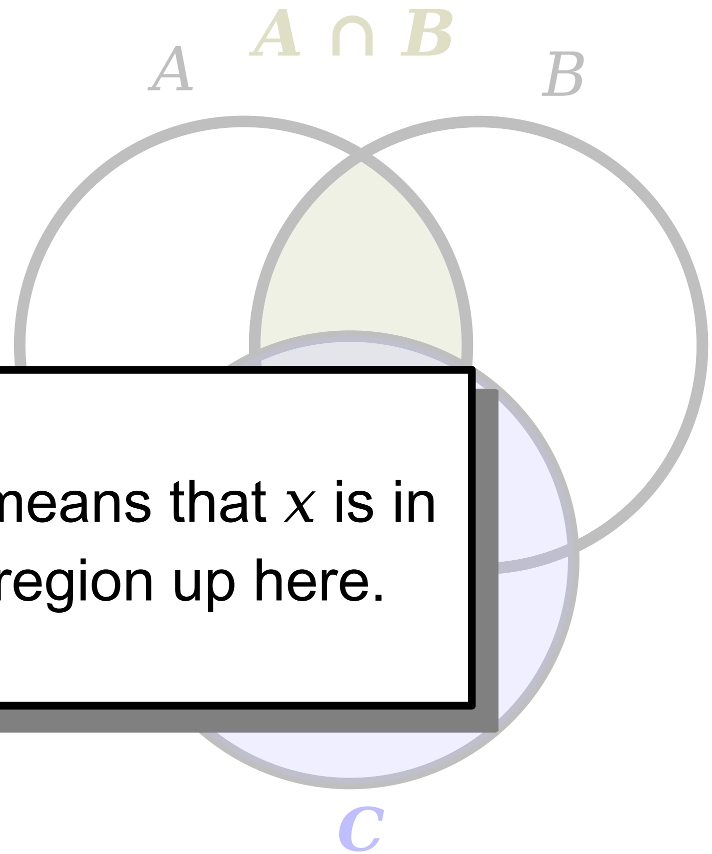


That means that x is in this region up here.



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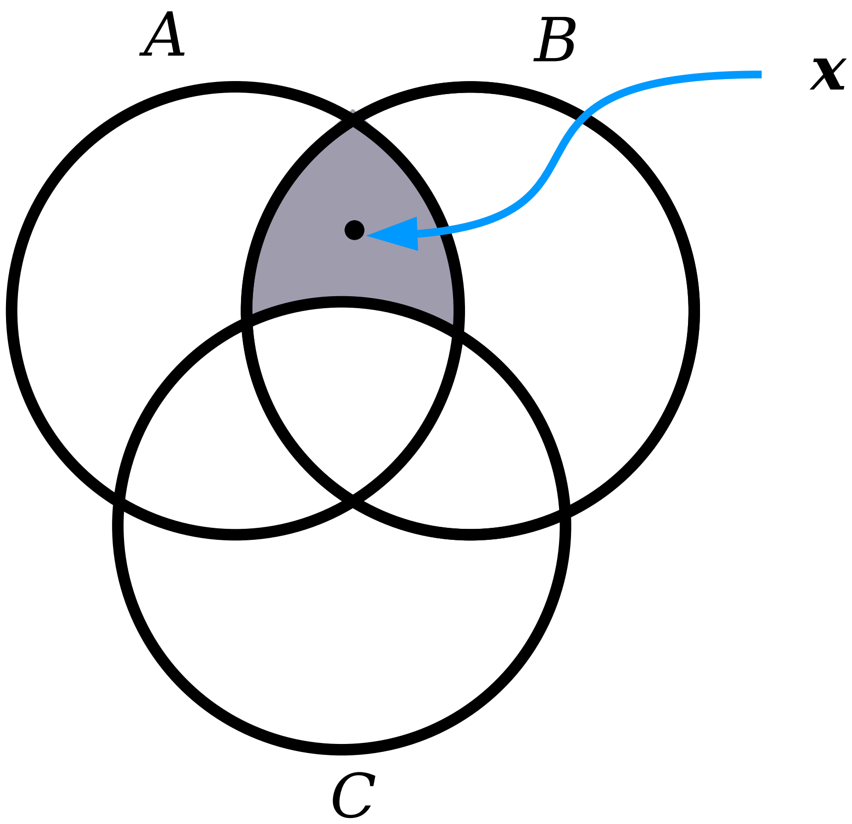

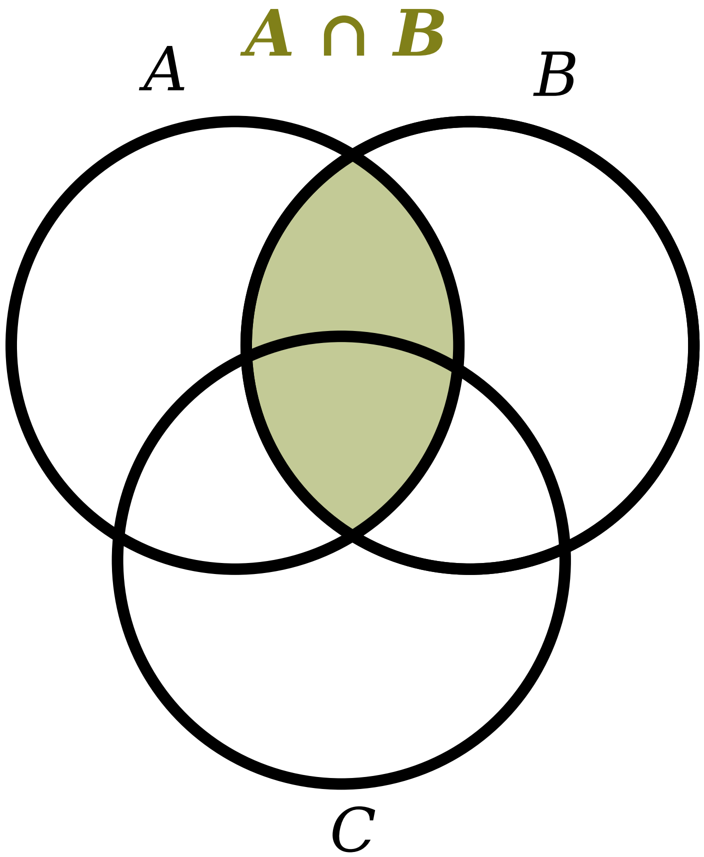

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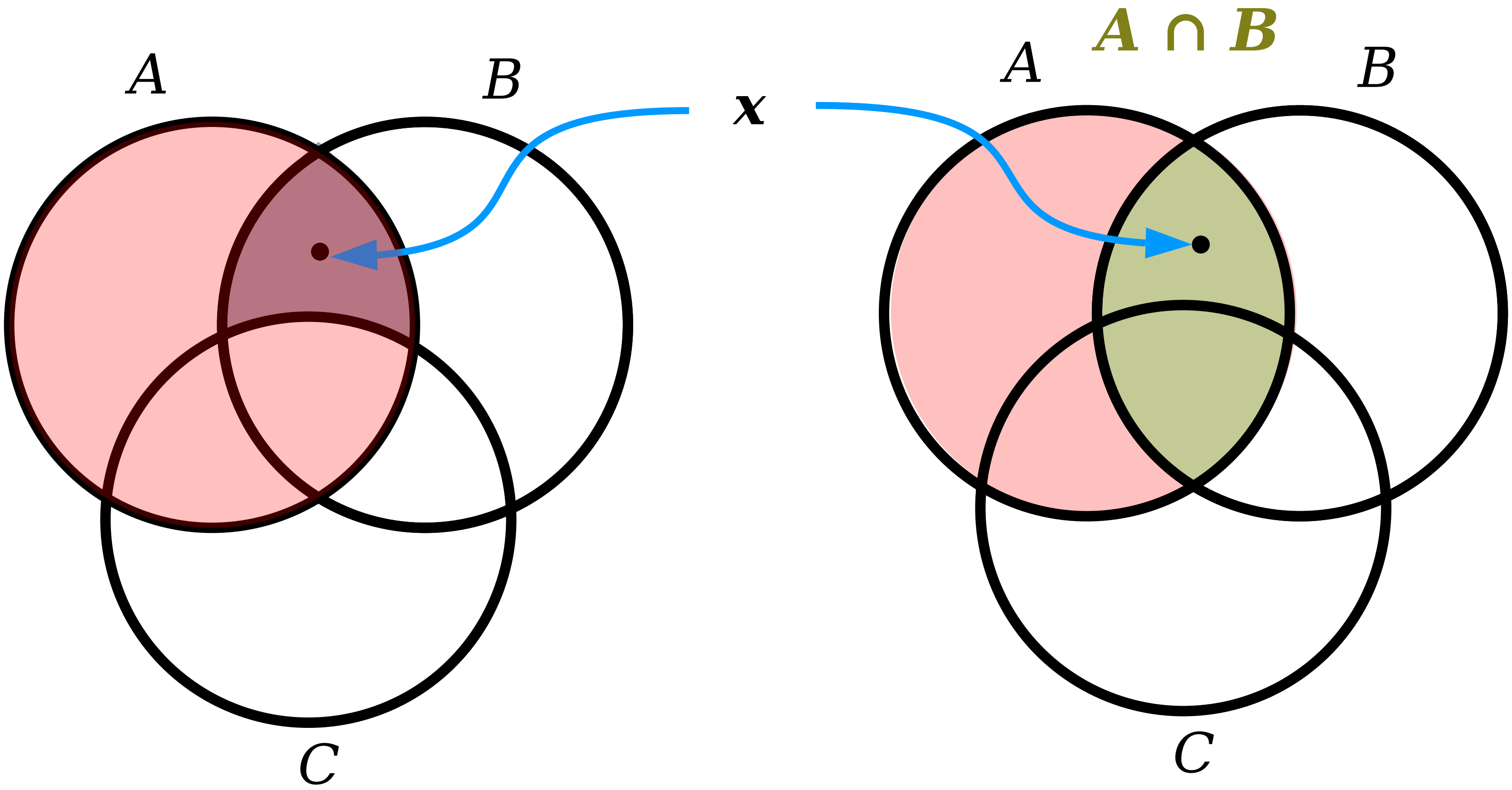
Let’s Draw Some Pictures!



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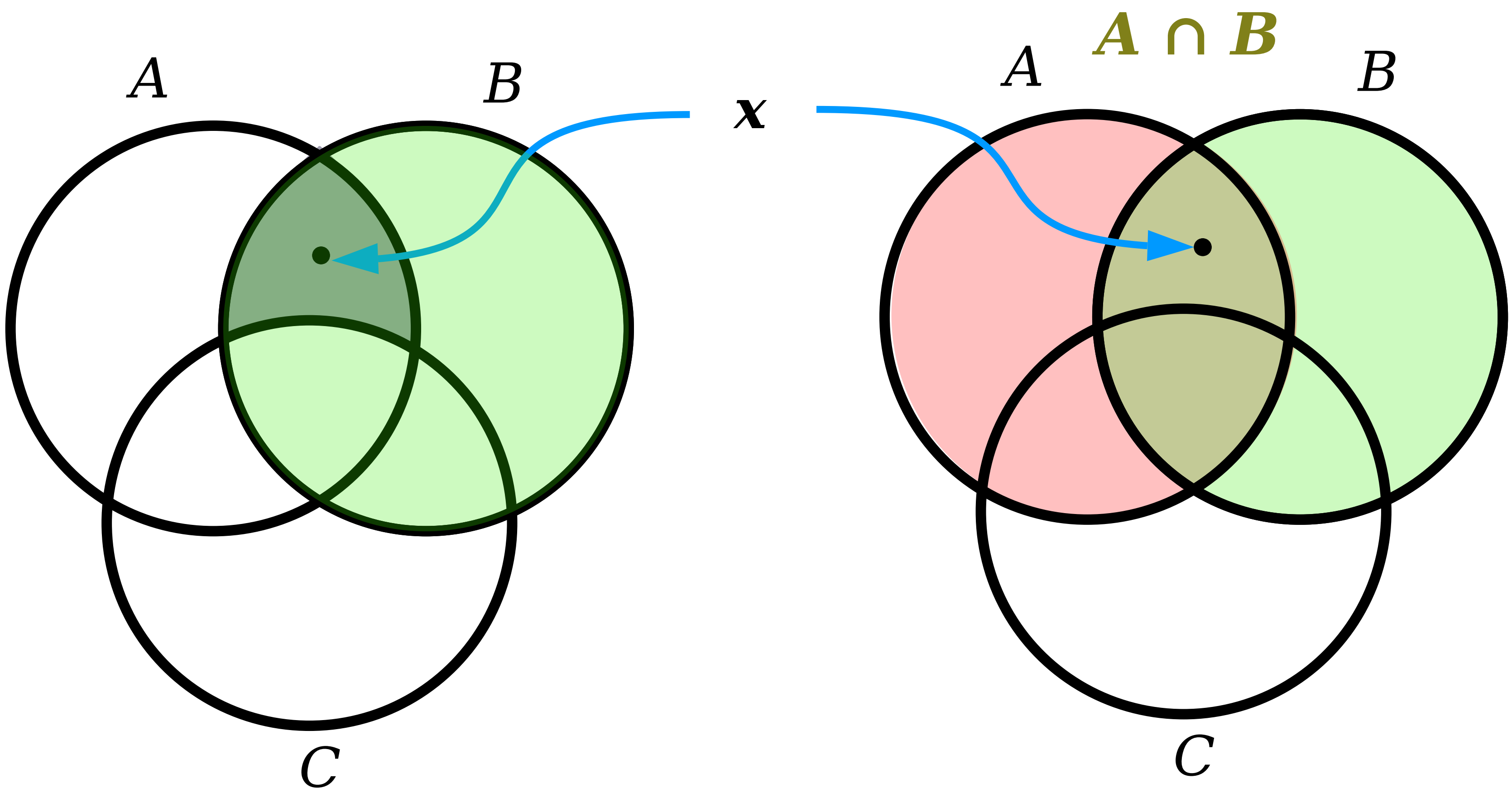

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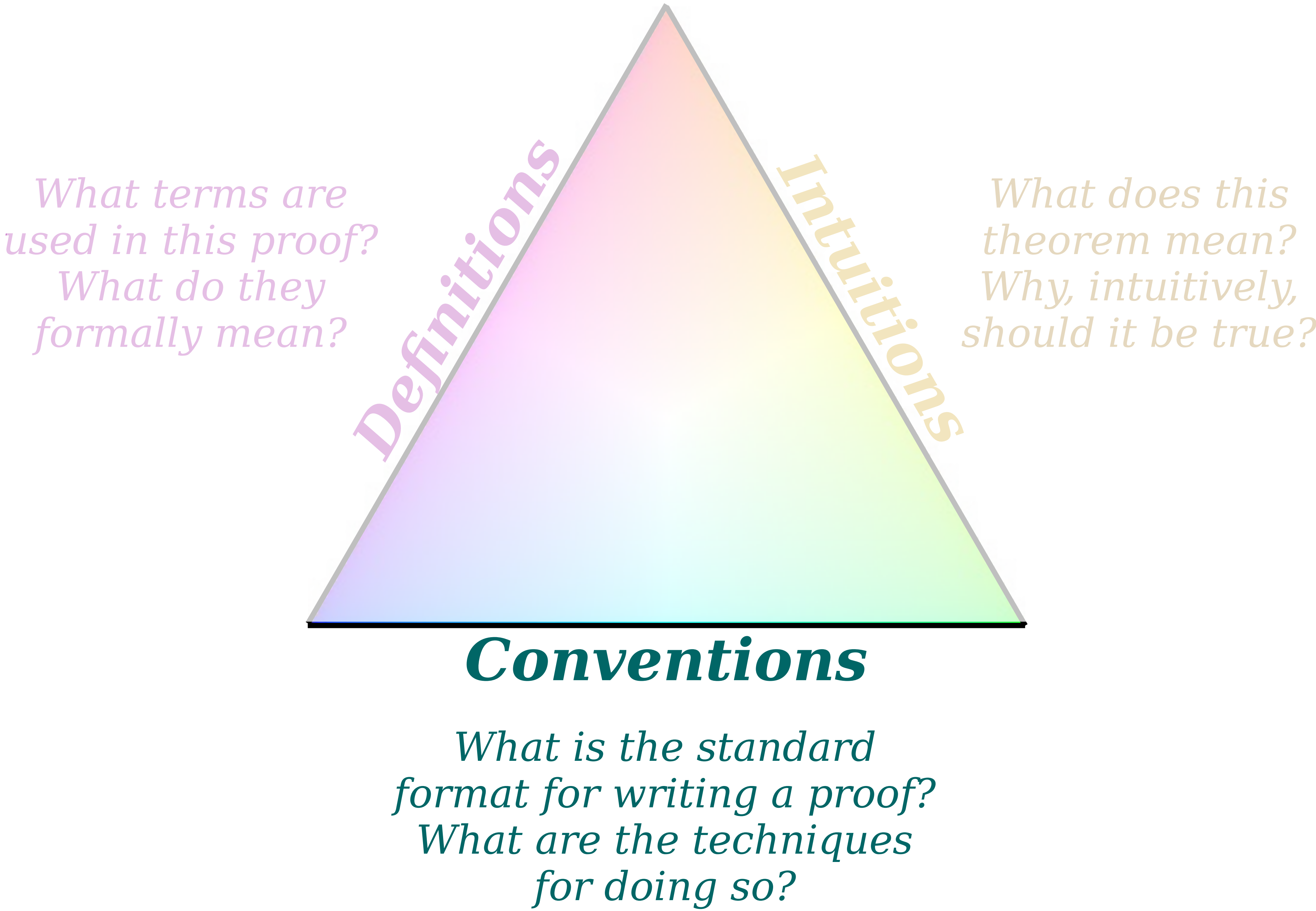
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Proof: Pick any sets A, B, and C. Then, choose any element
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Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

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Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$.

Case 2: $x \notin C$.

Theorem: If A, B, and C are sets, then

$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

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Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

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Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

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Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

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Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

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Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case … ll.

Case … $x \in A$

or … e're

lef … $B \cup C$

we …

These are ***arbitrary choices***. Rather than specifying what ***A***, ***B***, and ***C*** are, we're signaling to the reader that they could, in principle, supply any choices of ***A***, ***B***, and ***C*** that they'd like.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then

$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since x

that $x \in$

Case This means $x \in (A \cap B) \cup C$ as well.

Case

or

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we need

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

Notice that the statement of the theorem doesn't include any variable named *x*. We introduced this variable because that's what the definition says to do.

This is common in proofwriting. Always call back to the definition to make sure you're proving the right thing!

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any set
$x \in (A \cup C) \cap (B$

Since $x \in (A \cup C$
that $x \in B \cup C$. W

> As before, it's good to summarize what we established when splitting into cases.

Case 1: $\boldsymbol{x \in C}$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $\boldsymbol{x \notin C}$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

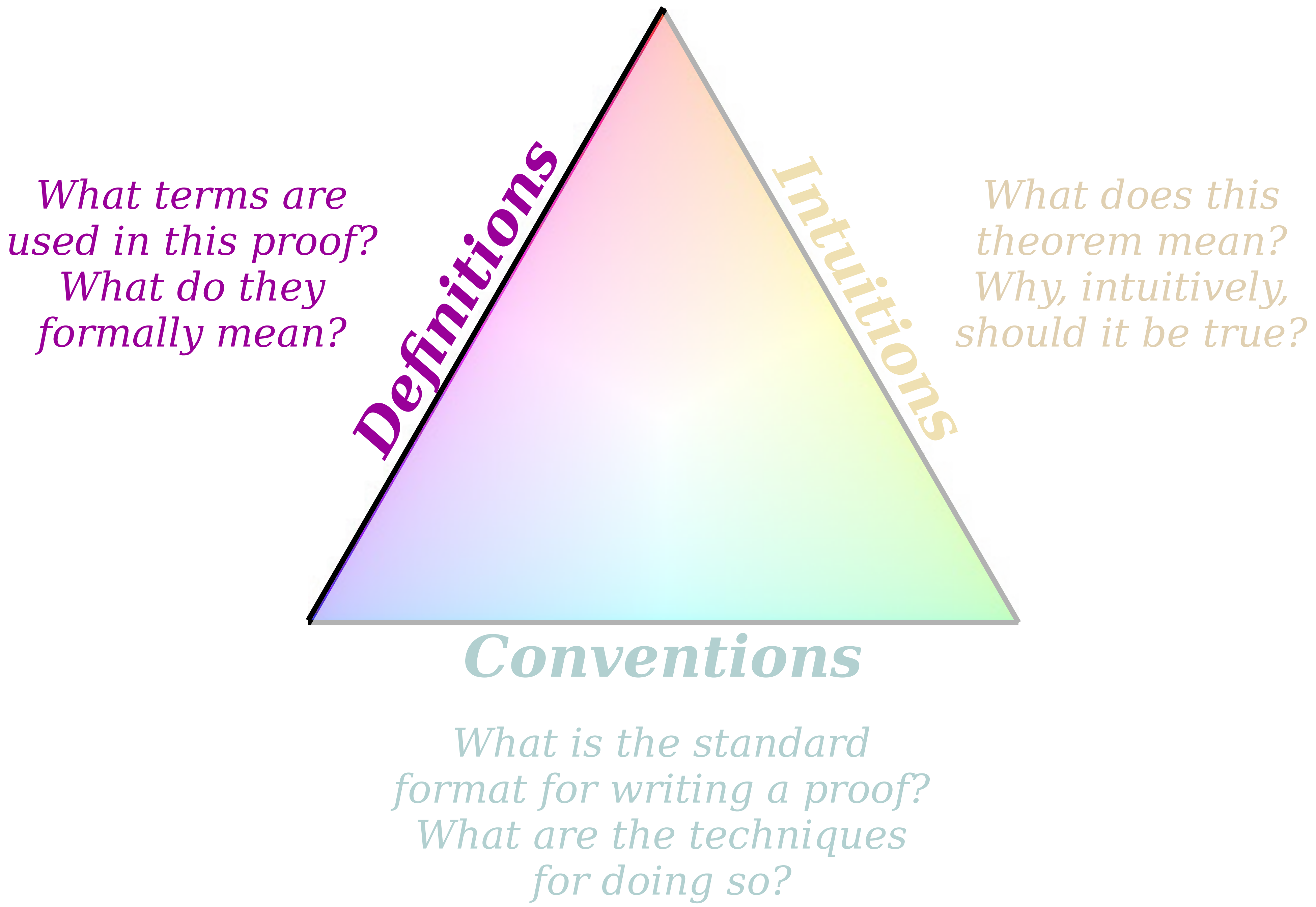
Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$.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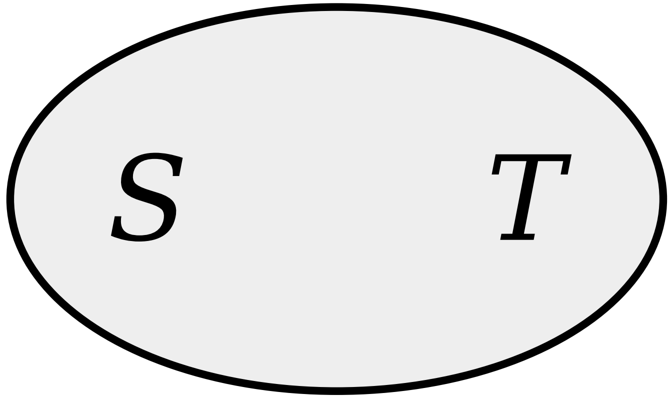
Set Equality



$S = T$

Definition: If S and T are sets, then $S = T$ if

$$S \subseteq T \quad \text{and} \quad T \subseteq S.$$

To prove that $S = T$:

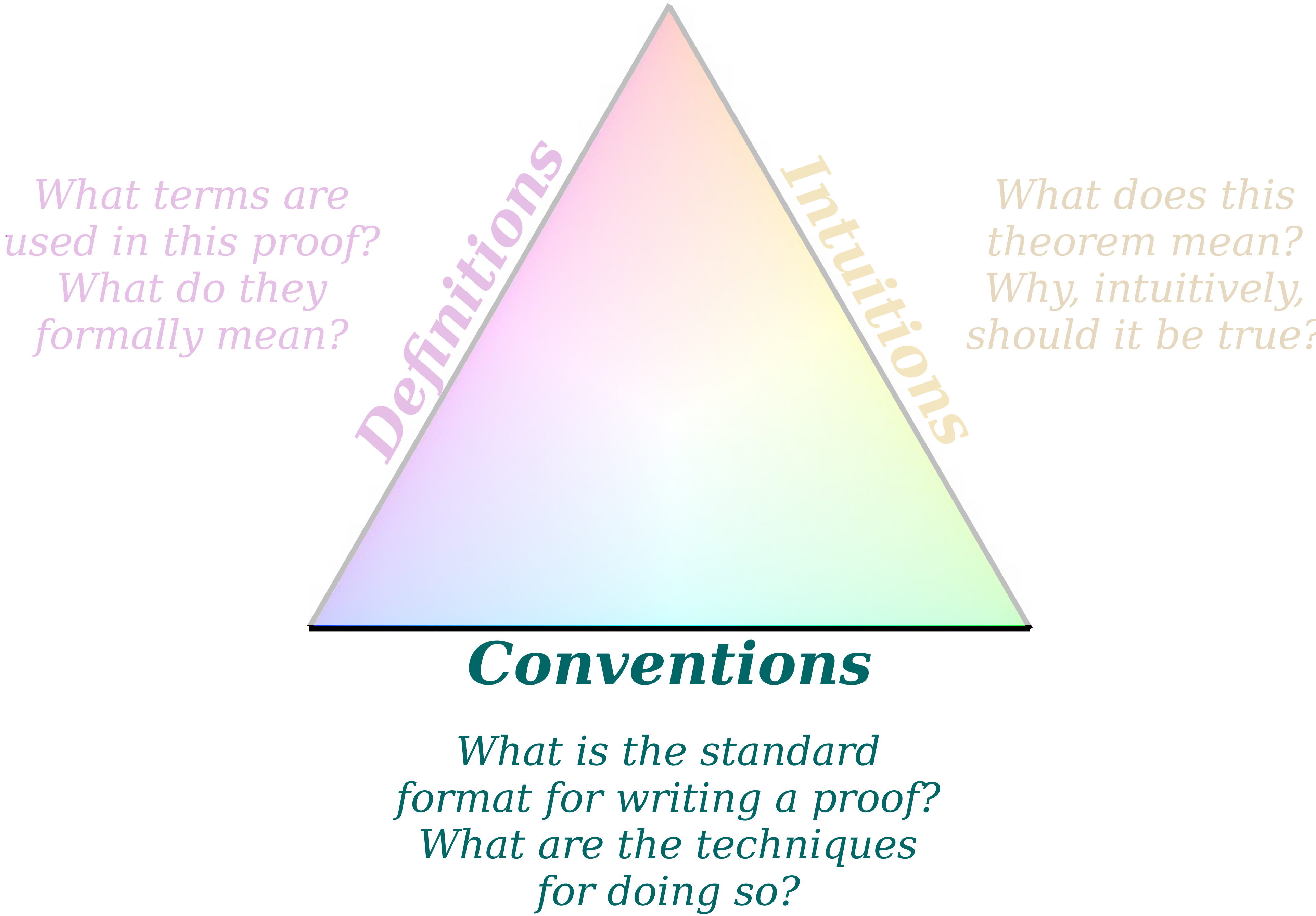
Prove that $S \subseteq T$ and $T \subseteq S$.

If you know that $S = T$:

If you have an $x \in S$, you can conclude $x \in T$.

If you have an $x \in T$, you can conclude $x \in S$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$.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Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$.

Proof: Fix any sets A, B, and C. We need to show that

$$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C \quad (1)$$

and that

$$(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C). \quad (2)$$

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Since both (1) and (2) hold, we know that each of these two sets are subsets of one another, and therefore that the sets are equal.

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$(A \cup C) \cap (B \cup C) =$

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$(A \cup C) \cap (B \cup$

and that

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It is ***common*** for proofs in math to build on one another. That's how we make progress and make new discoveries!

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Let’s take a quick break!

Indirect Proofs

Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Logical Implication

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the ***antecedent***.

This part of the implication is called the ***consequent***.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

What Implications Mean

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
  - Rainbows do not cause rain.

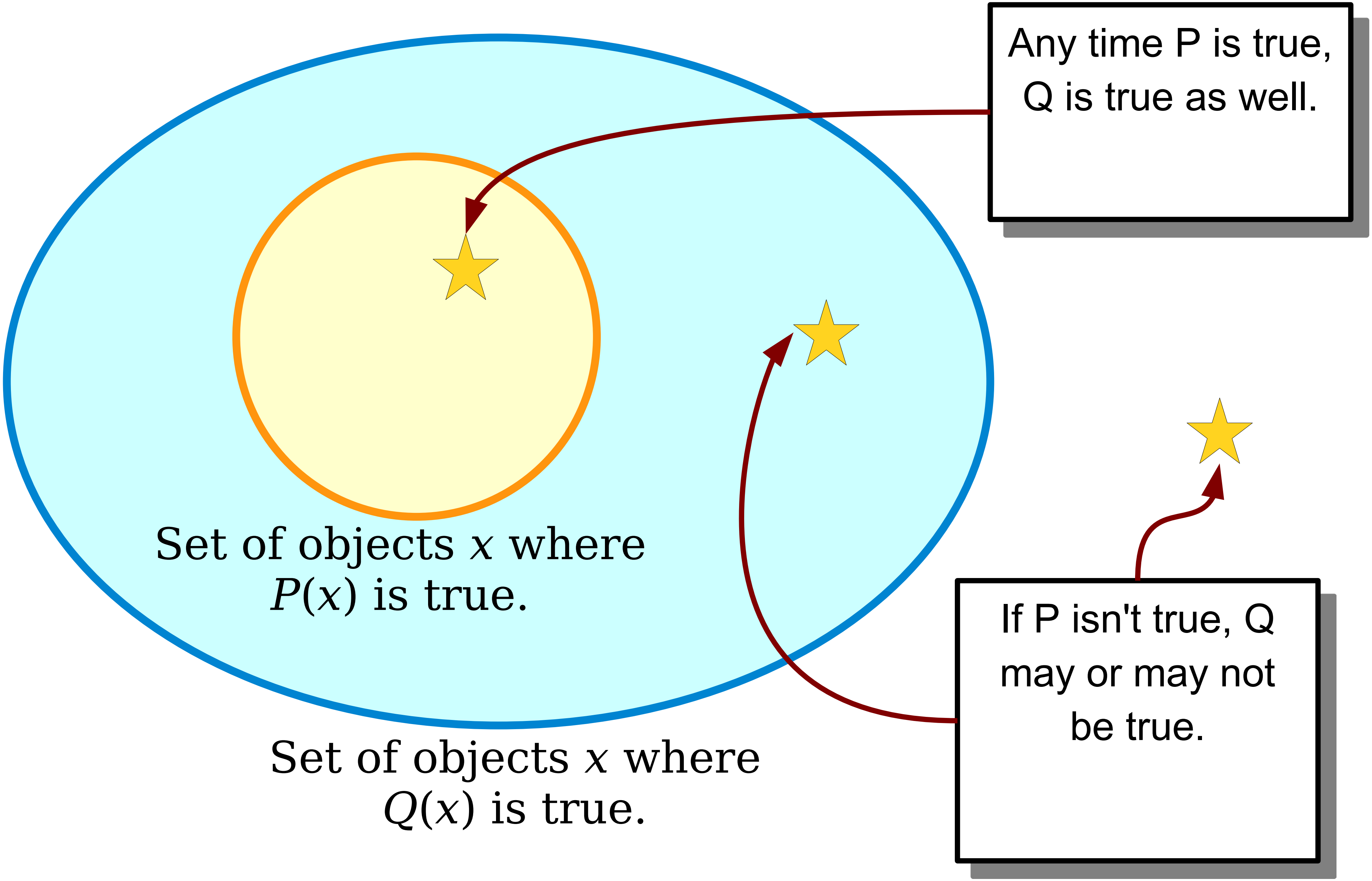
What Implications Mean

- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



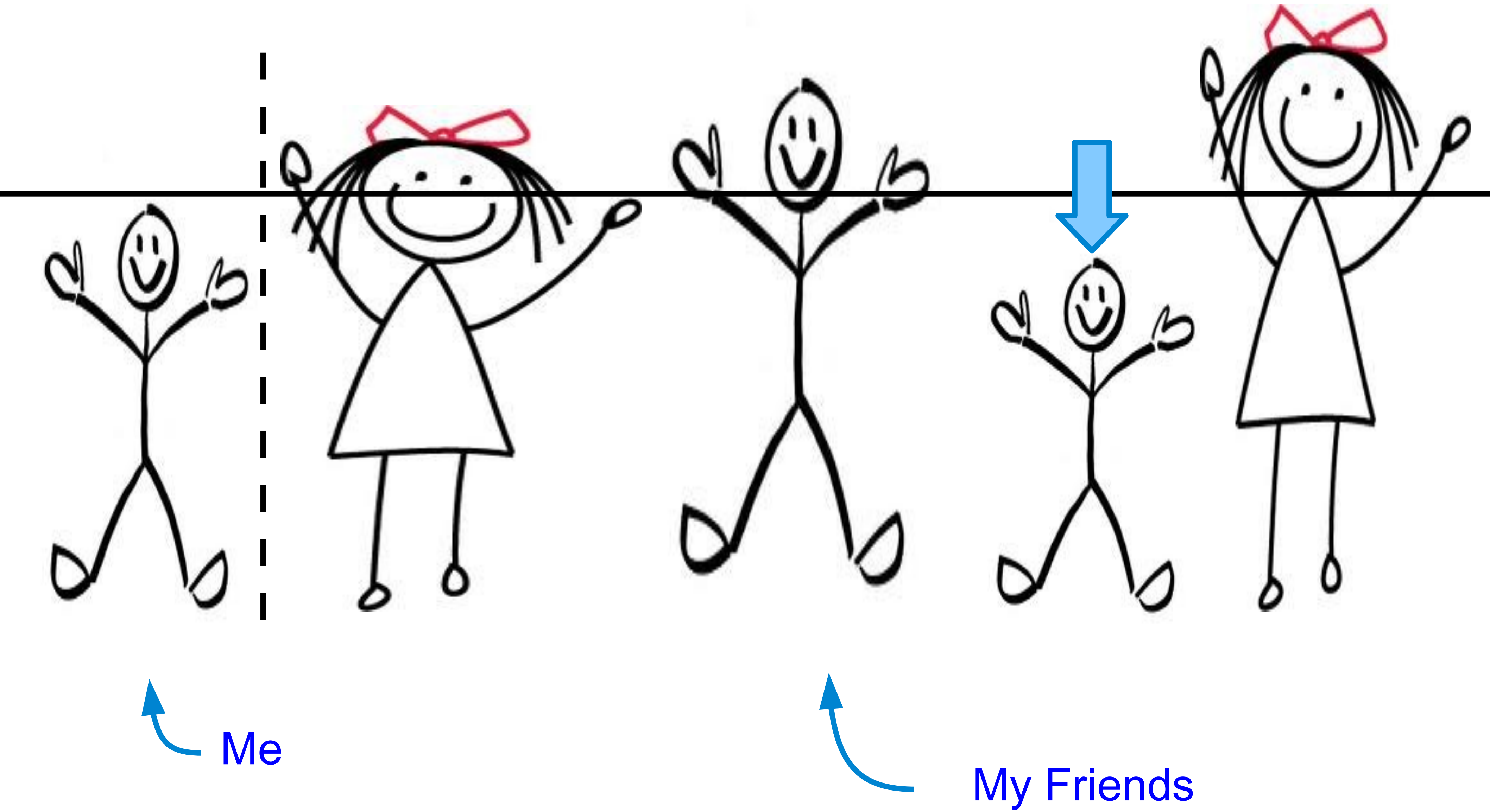
Negations

Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the proposition "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠
  - Its negation is *not* "we're in the Bay Area." ⚠

How do you find the negation of a statement?

"All My Friends Are Taller Than Me"



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

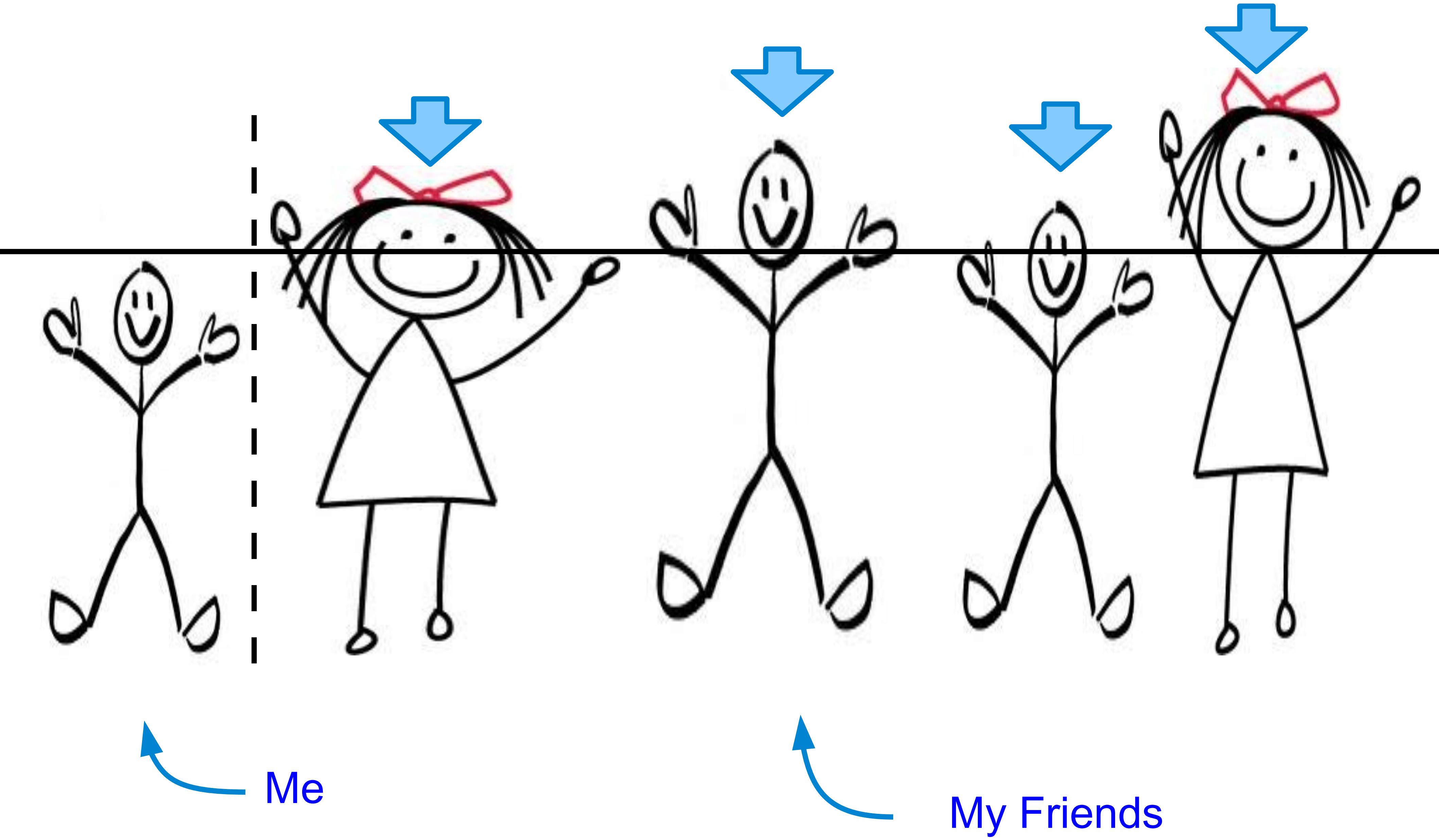
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the ***universal*** statement

For all *x*, *P*(*x*) is false.

How do you negate an implication?


Story Time!

Ancient Contract:

If Nanni pays money to Ea-Nasir, then
Ea-Nasir will give Nanni quality copper ingots.

Ancient Contract:

If Nanni pays money to Ea-Nasir, then Ea-Nasir will give Nanni quality copper ingots.

Question: What has to happen for the contract to be broken?

A) Nanni does not pay Ea-Nasir and Ea-Nasir does not give the ingots.

B) Nanni does not pay Ea-Nasir and Ea-Nasir gives the ingots.

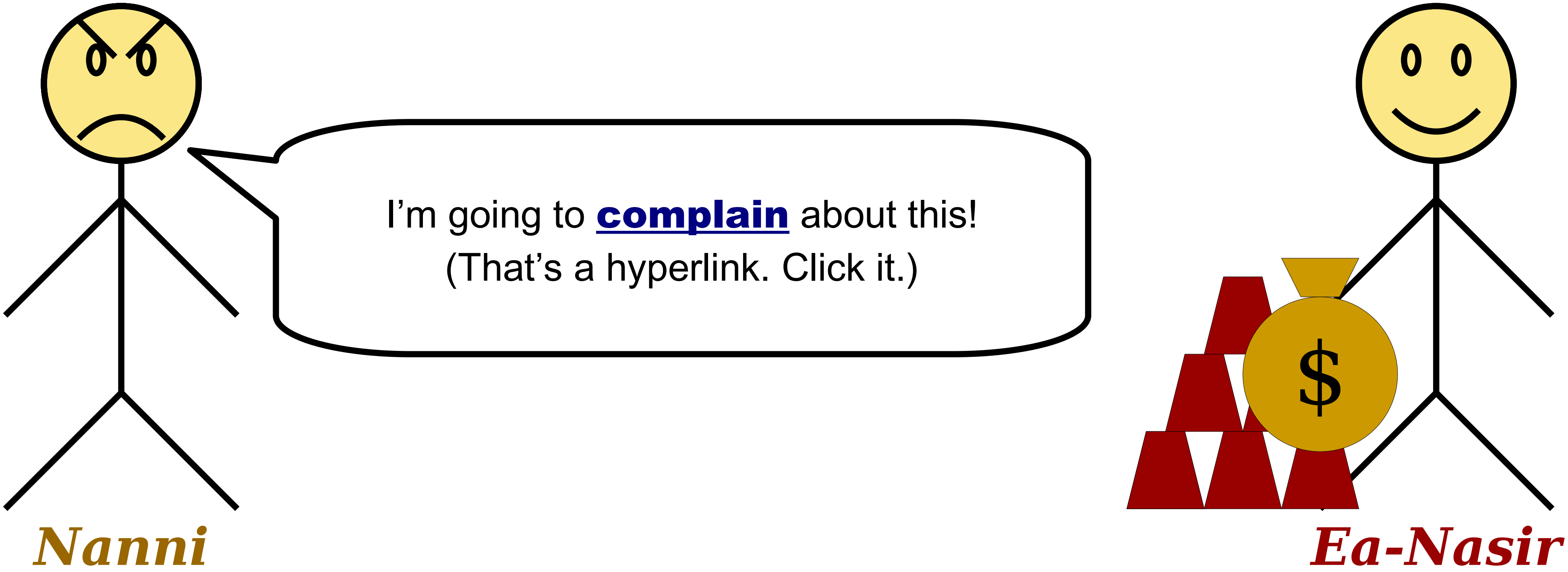
C) Nanni pays Ea-Nasir and Ea-Nasir does not give the ingots.

D) Nanni pays Ea-Nasir and Ea-Nasir gives the ingots.

Respond at pollev.com/robynreiss

Ancient Contract:

If Nanni pays money to Ea-Nasir, then
Ea-Nasir will give Nanni quality copper ingots.



Question: What has to happen for this contract to be broken?
Answer: Nanni pays Ea-Nasir and doesn't get quality copper ingots.

The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!

The negation of the statement

“For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”

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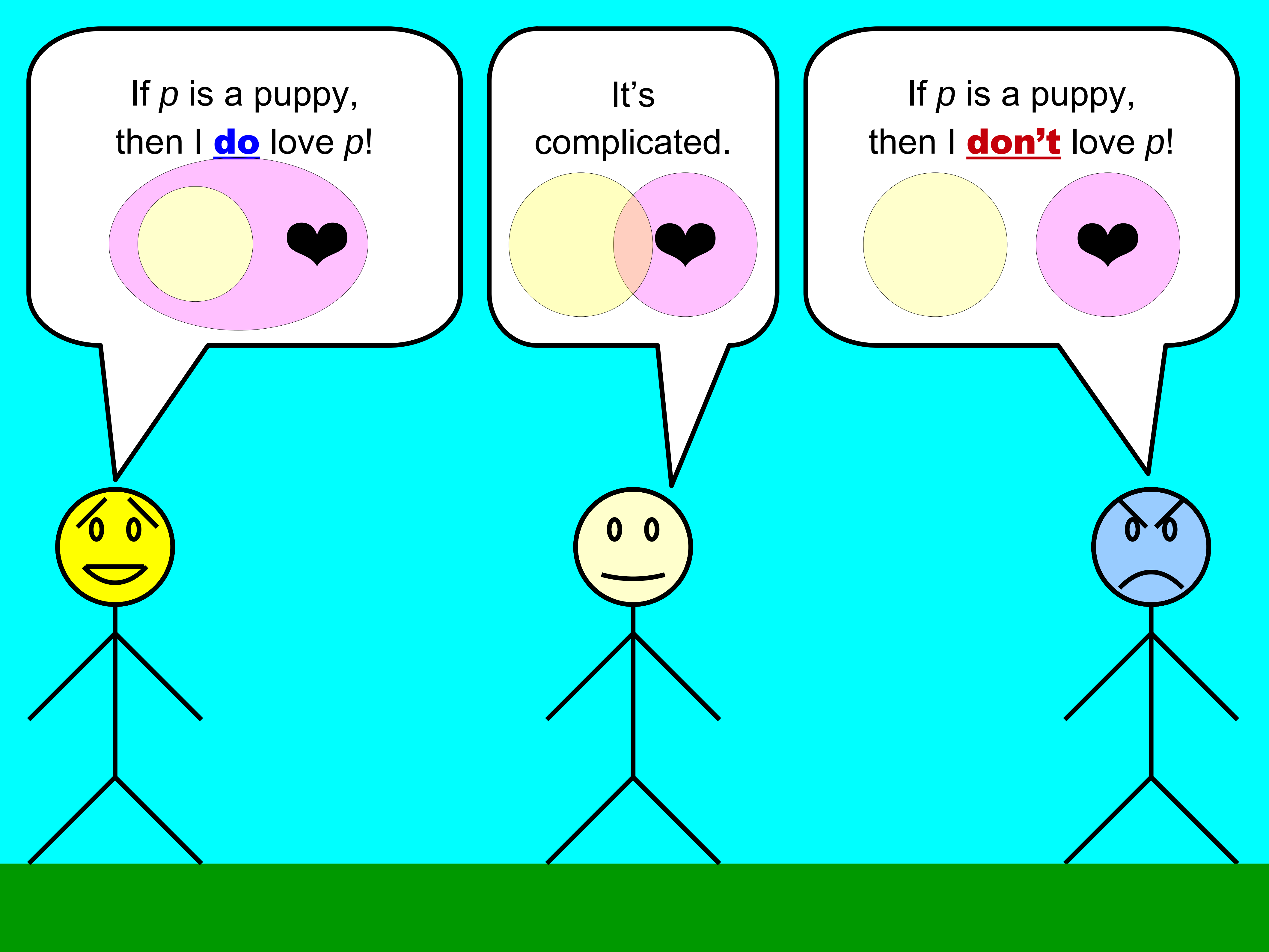
The negation of an implication is not an implication!


If *p* is a puppy, then I **do** love *p*!
It's complicated.
If *p* is a puppy, then I **don't** love *p*!

How to Negate Universal Statements:

"For all *x*, *P*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is false."

How to Negate Existential Statements:

"There exists an *x* where *P*(*x*) is true"

becomes

"For all *x*, *P*(*x*) is false."

How to Negate Implications:

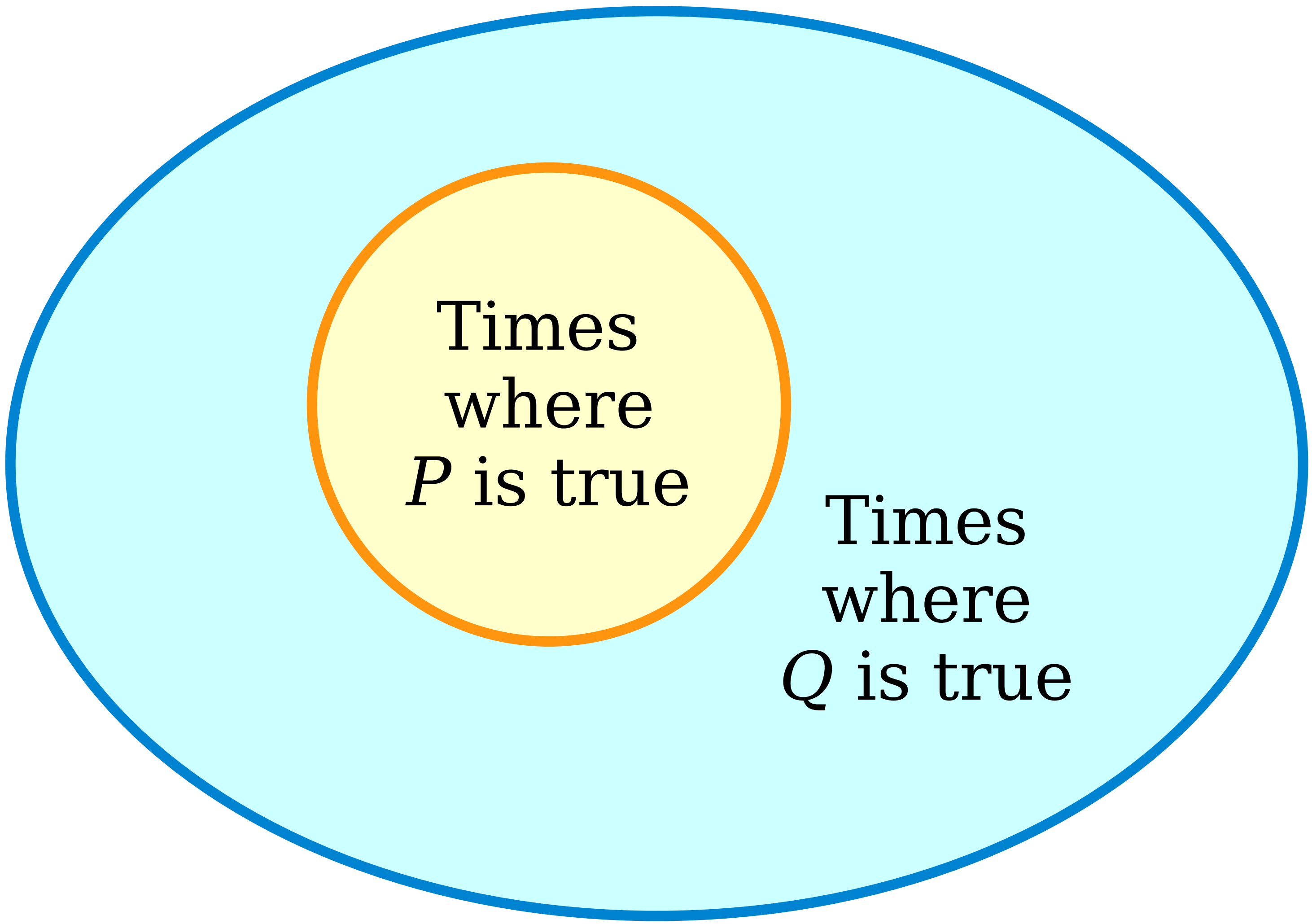
"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

becomes

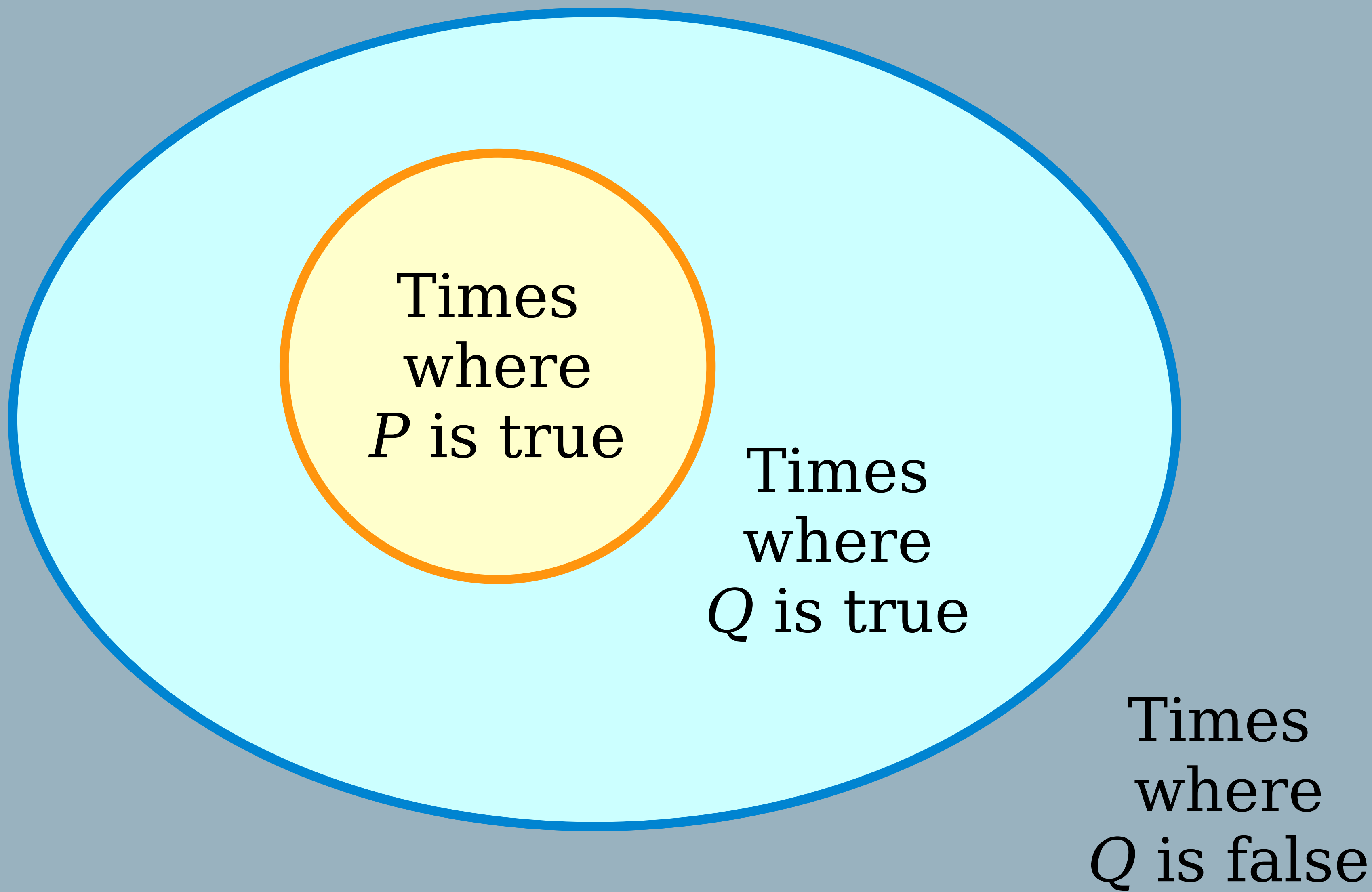
"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false."

Proof by Contrapositive

Implication, Diagrammatically



Implication, Diagrammatically



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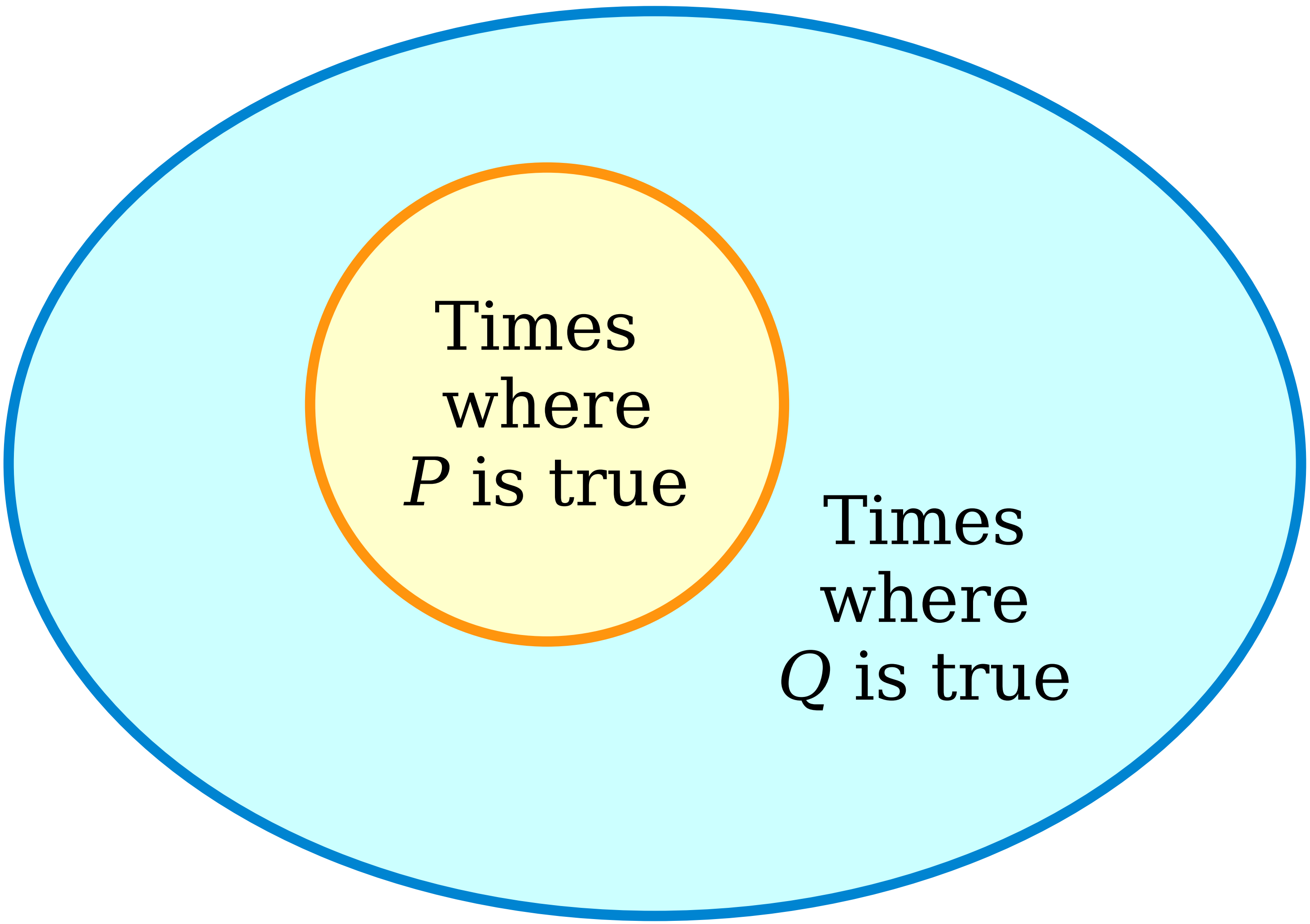
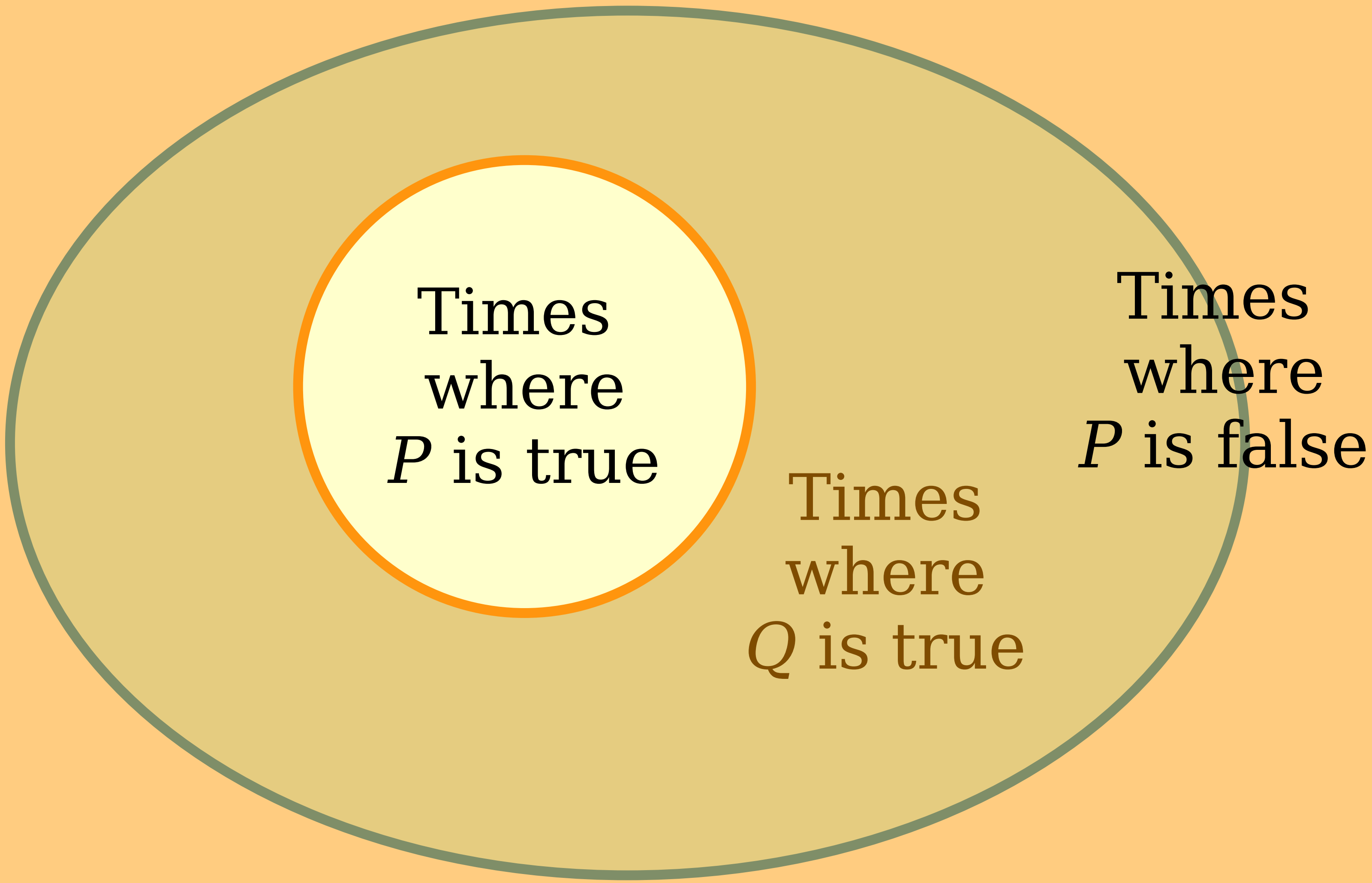
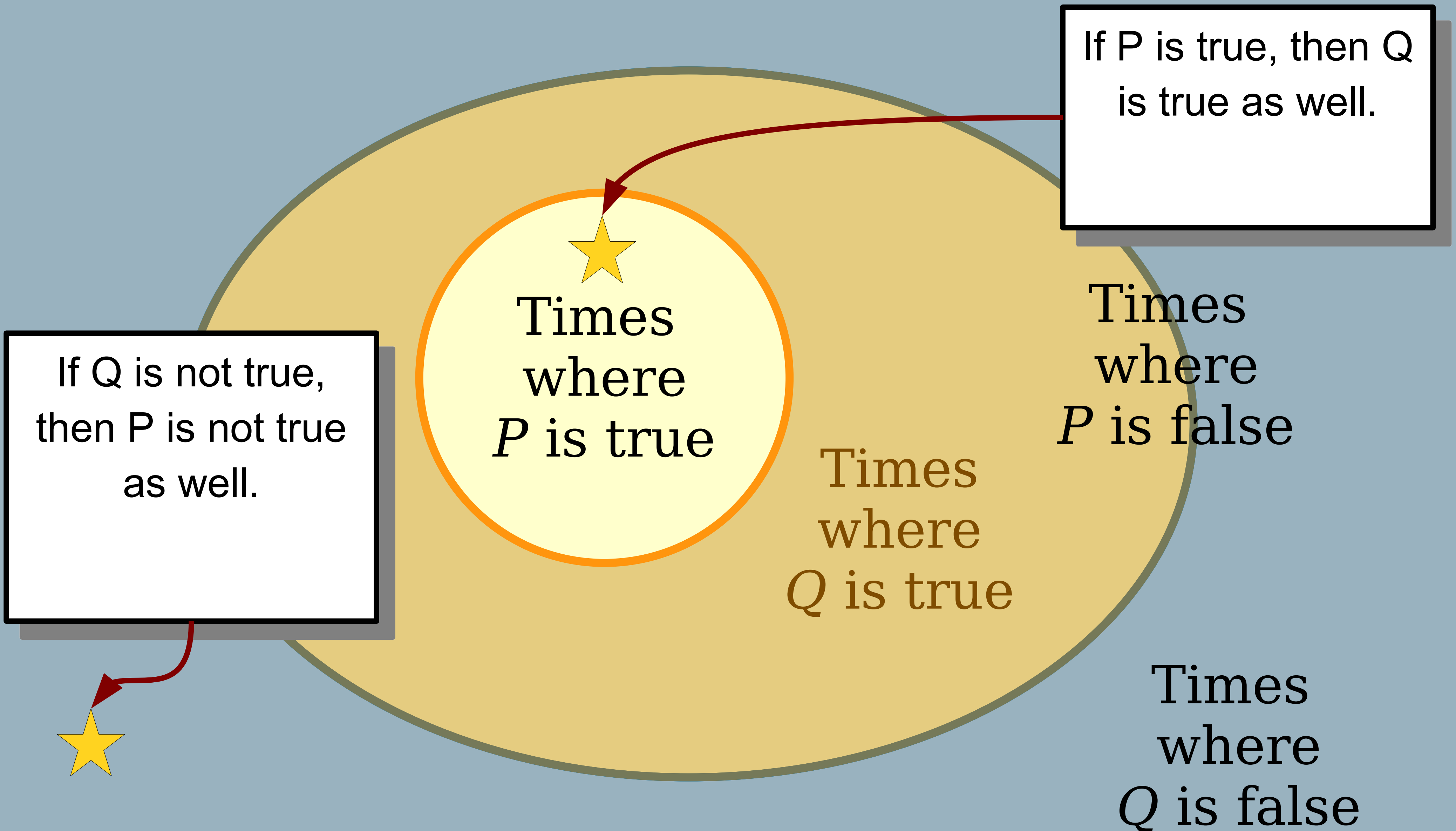


Implication, Diagrammatically



Implication, Diagrammatically



If P is true, then Q is true.

If Q is false, then P is false.

What are the negations of the above two statements?

If P is true, then Q is true.

↕ *negates to*

P is true and Q is false.

If Q is false, then P is false.

What are the negations of the above two statements?

If P is true, then Q is true.

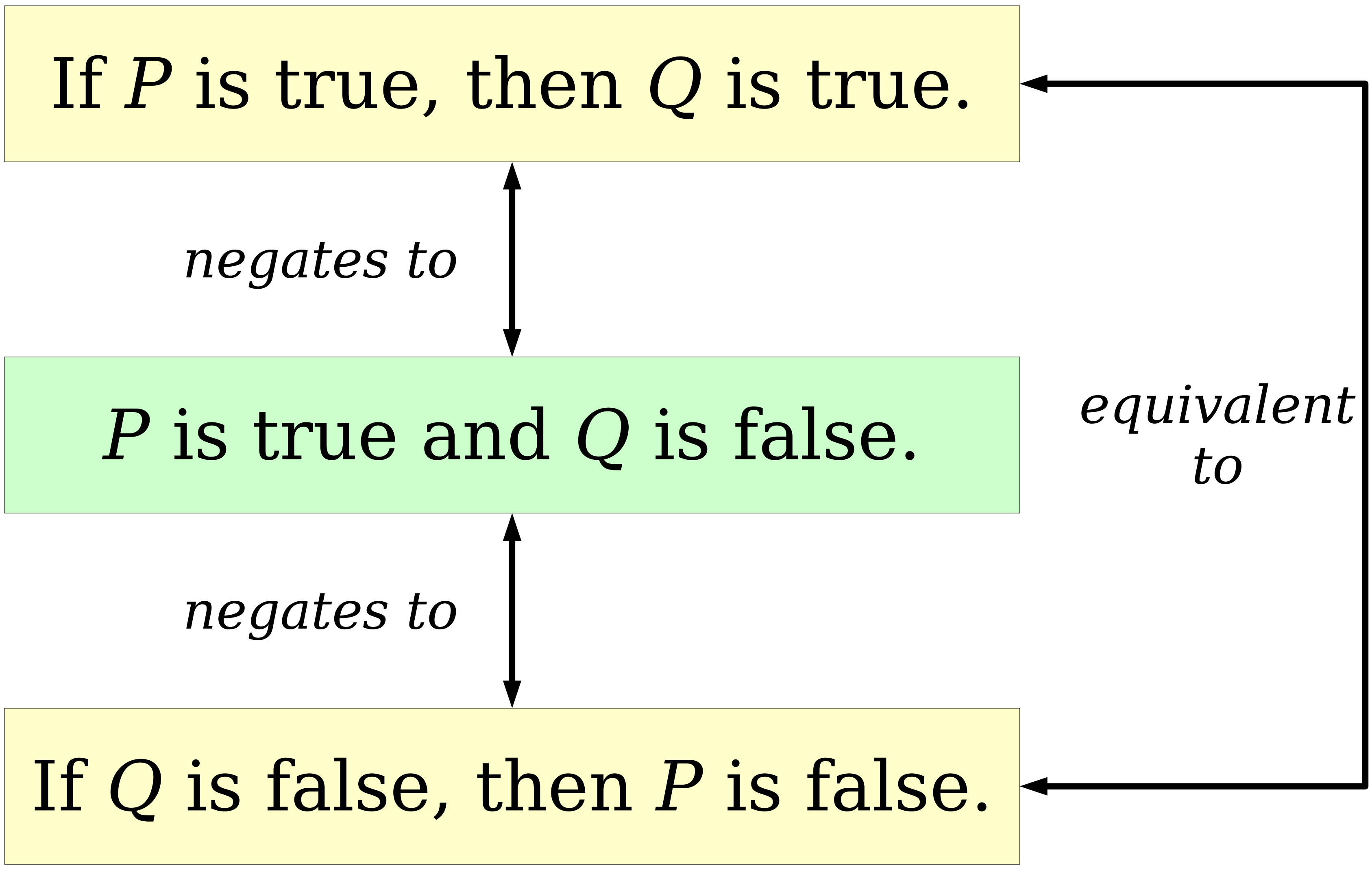
negates to

P is true and Q is false.

negates to

If Q is false, then P is false.

What are the negations of the above two statements?



What are the negations of the above two statements?

The Contrapositive

- The ***contrapositive*** of the implication

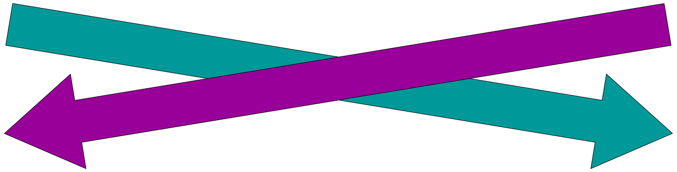
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

The Contrapositive

- The ***contrapositive*** of the implication

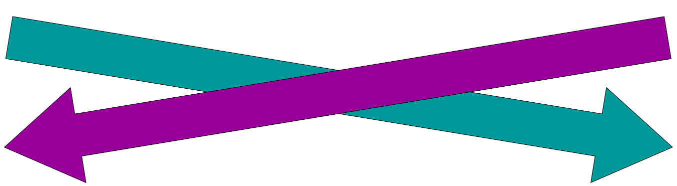
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won't steal it.



If raccoons stole the cat food, then I didn't store it inside.

To prove the statement

"if *P* is true, then *Q* is true,"

you can choose to instead prove the equivalent statement

"if *Q* is false, then *P* is false,"

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

This is a courtesy to the reader and says “heads up! we’re not going to do a regular old-fashioned direct proof here.”

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

Question: What is the contrapositive of this statement?

A) If n^2 is odd, then n is odd.

B) If n is odd, then n^2 is odd.

C) If n is even, then n^2 is even.

D) None of the above.

Respond at pollev.com/robynreiss

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

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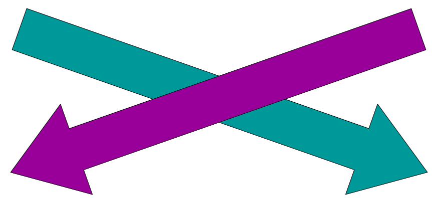

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

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We know ... integer ... us that

From th... (namely ... means t... to show. ■

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer *n*, if *n* is even, then n^2 is even.

For any integer *n*, *i*f n^2 is even, then *n* is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

for any integer *n*: *n* is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

  ***P* if and only if *Q*,**

  you need to prove two separate statements.
  - First, that if *P* is true, then *Q* is true.
  - Second, that if *Q* is true, then *P* is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Let’s take a quick break!

Time-Out for Announcements!

Outdoor Activities Guide

- Being on campus means you're less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

  https://cs103.stanford.edu/outdoor_activities
- A sampler of what to check out:
  - Drive to the observatory in the mountains near San Jose and take in the views.
  - Visit a beach with an enormous colony of elephant seals.
  - Walk in redwood forests and pick your own bay leaves.


California is breathtaking!!



Readings for Today

- On the course website we have some information you should look over.
- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.

Problem Set 1

- Problem Set 0 is due at 5:30PM tomorrow.
  - Missed the deadline? Ping us and we’ll see what we can do.
- Problem Set 1 goes out today. It’s due next Thursday at 5:30PM.
  - Explore the language of set theory and better intuit how it works.
  - Learn more about the structure of mathematical proofs.
  - Write your first “freehand” proofs based on your experiences.
- As always, reach out if you have any questions!

Submitting Assignments

- All assignments should be submitted through GradeScope.
  - The programming portion of the assignment gets submitted separately from the written component.
  - The written component ***must*** be typed up; handwritten solutions don't scan well and get mangled in GradeScope.
- We don't do late days in CS103. Because submission times are recorded automatically, we're strict about the submission deadlines.
  - ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
  - ***Very bad idea:*** Wait until the last minute to submit.
- However, we are pretty generous with how we grade. Your score on the problem sets is the square root of your raw score. So an 81% maps to a 90%, a 50% maps to a 71%, etc. This gives a huge boost even if you need to turn something in that isn't done.

Getting Help

- It is ***completely normal*** in this class to need to get help from time to time.
- Feel free to ask clarifying and conceptual questions on EdStem.
- Need more structured help? We have office hours! Feel free to stop on by.
  - Check out the online “Guide to Office Hours” for more information about how our office hours system works.
  - The OH calendar is available on the course website.

Working in Pairs

- You can work on problem sets individually or in pairs.
- Each person/pair should only submit a single problem set. In other words, if you’re working in a pair, you and your partner should agree who will make the submission.
- For more details, check the Syllabus and Honor Code pages on the course website.

Finding a Problem Set Partner

Looking for a problem set partner?

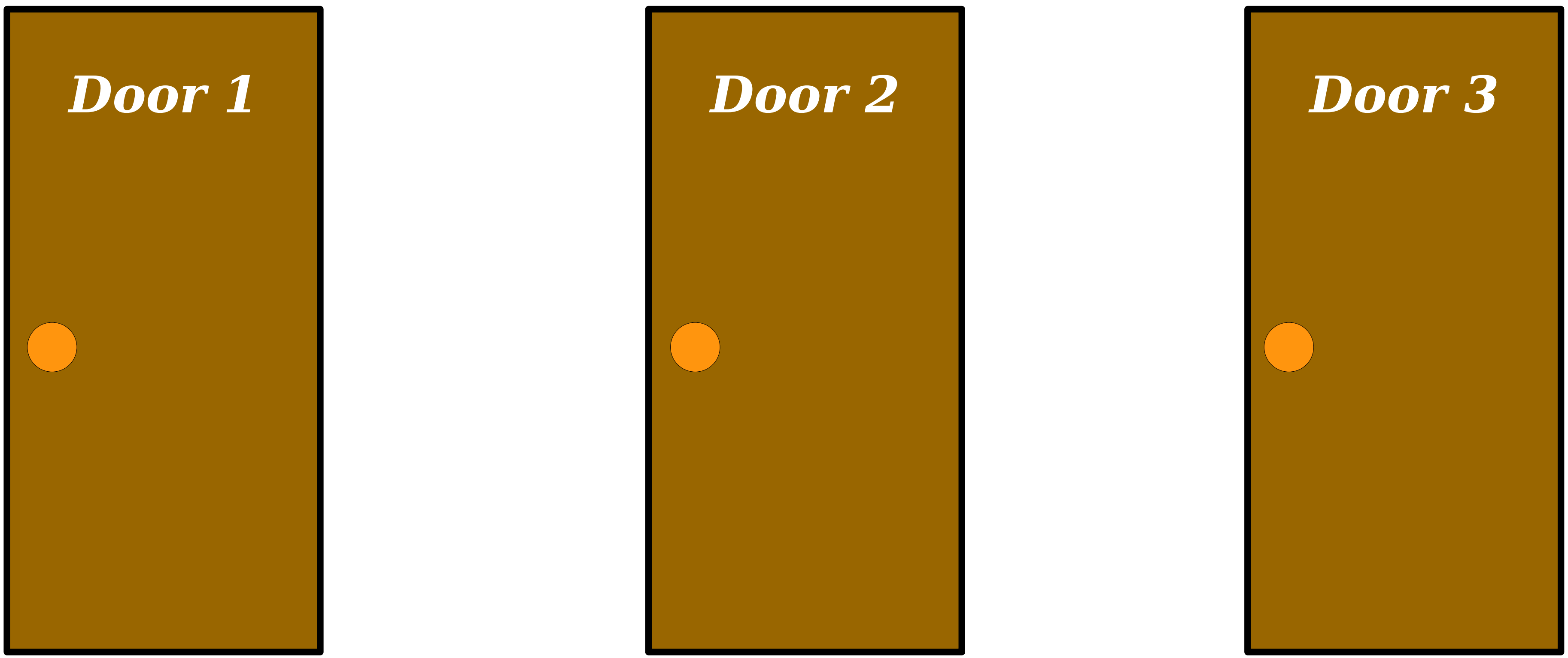
- Meet folks in lecture!
- Meet folks in office hours!
- Check out our pinned thread on EdStem!
- Fill out our matchmaking form!

A Note on the Honor Code

Back to CS103!

Proof by Contradiction

There's something hidden behind one of these doors. Which door is it hidden behind?



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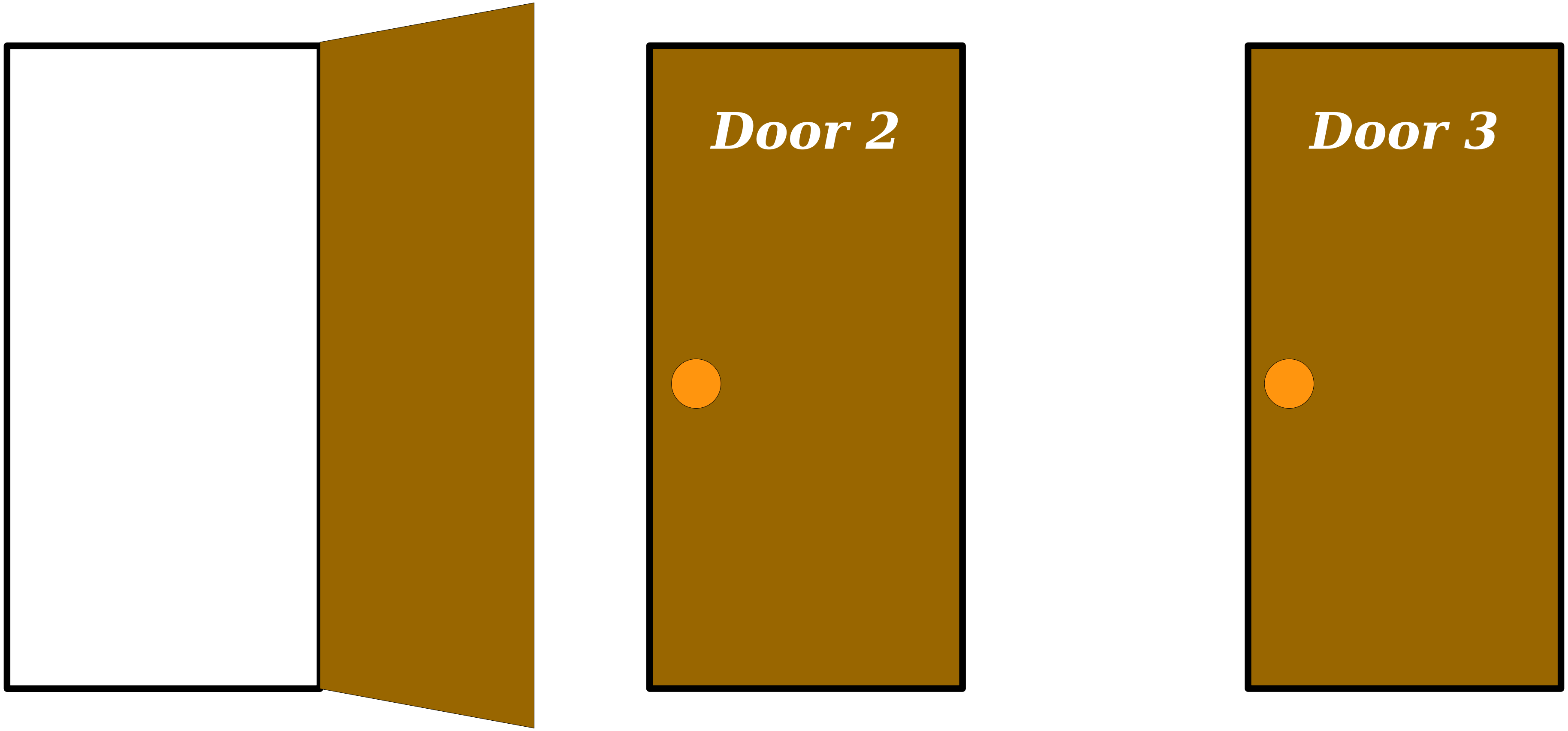

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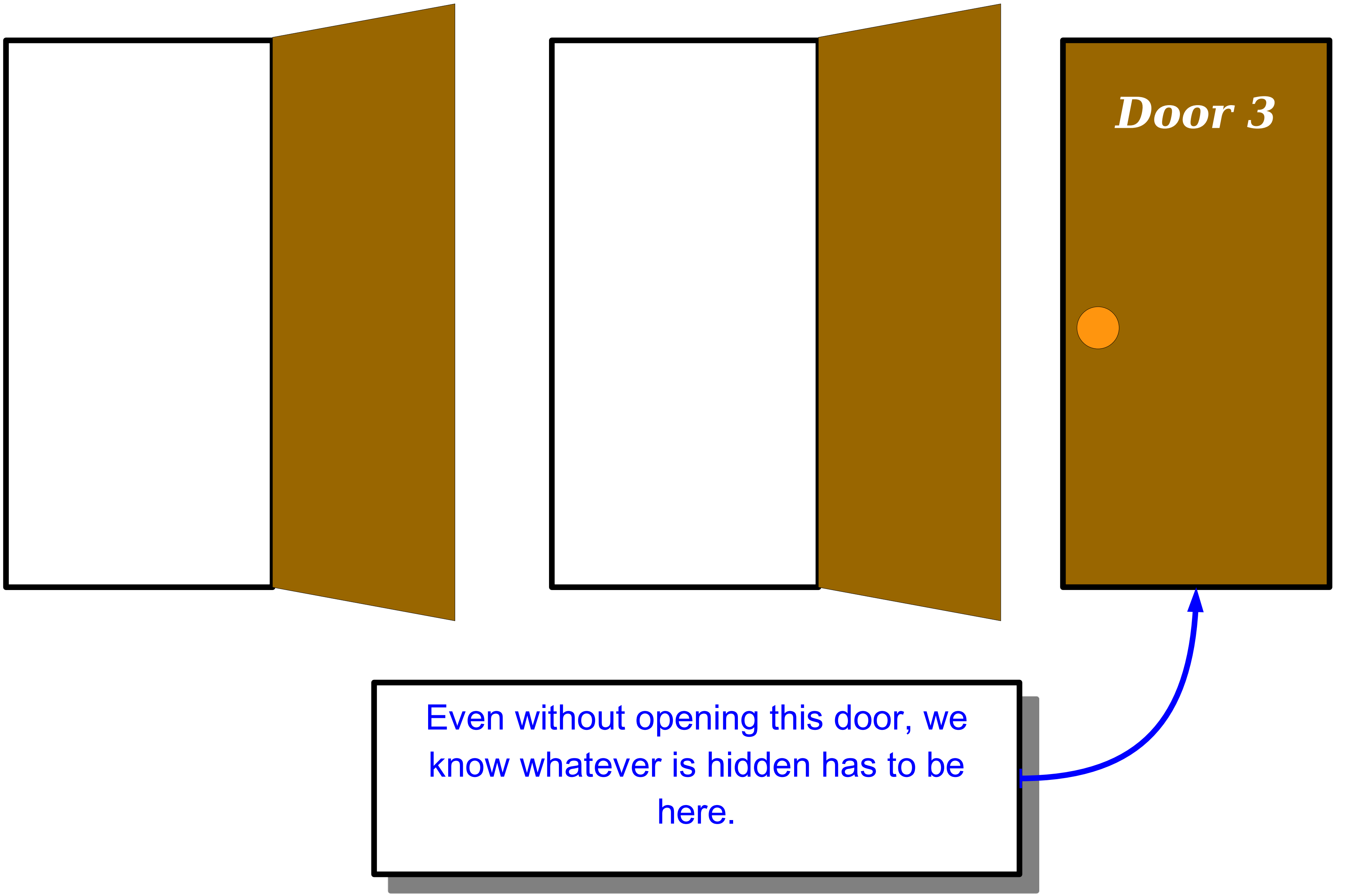

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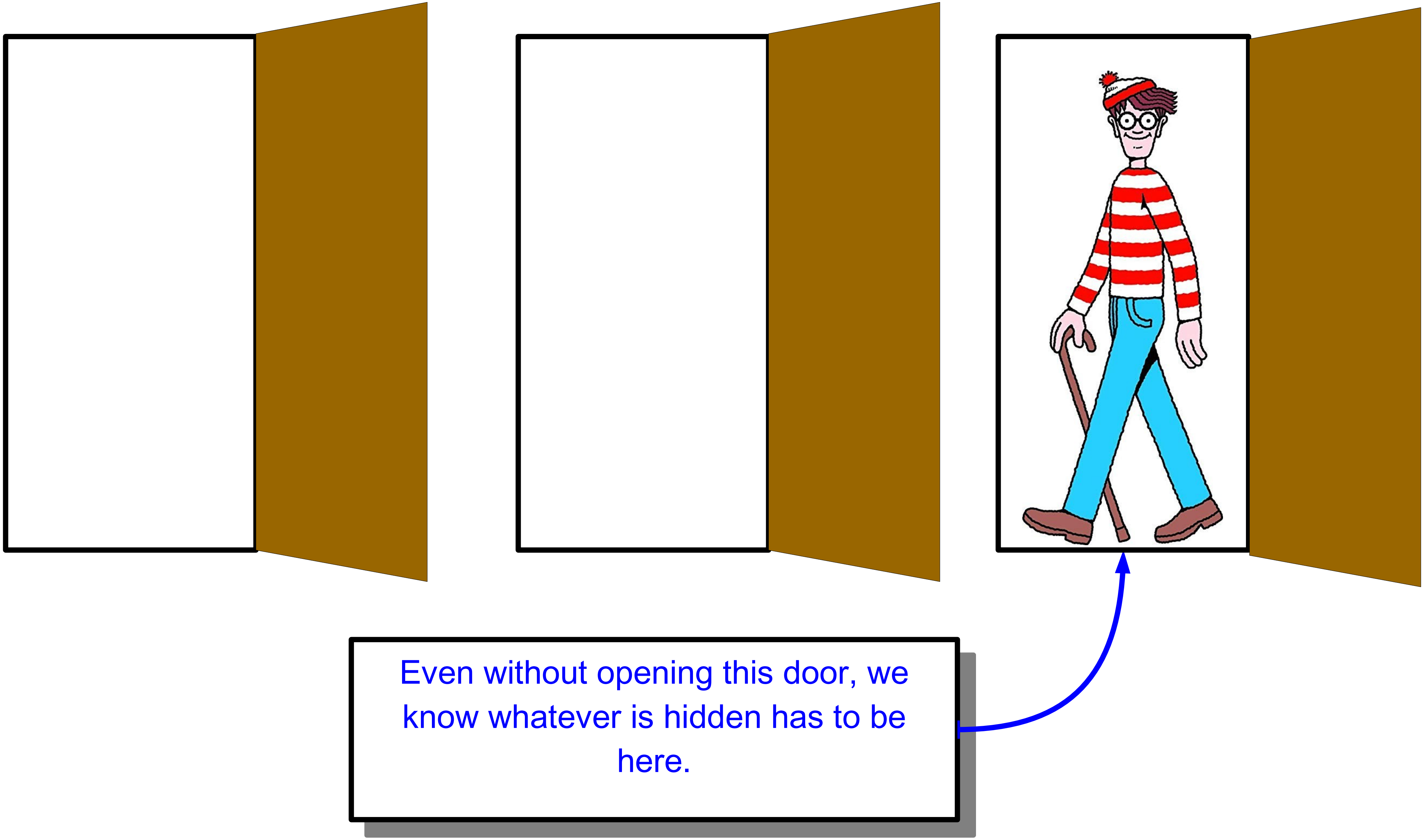
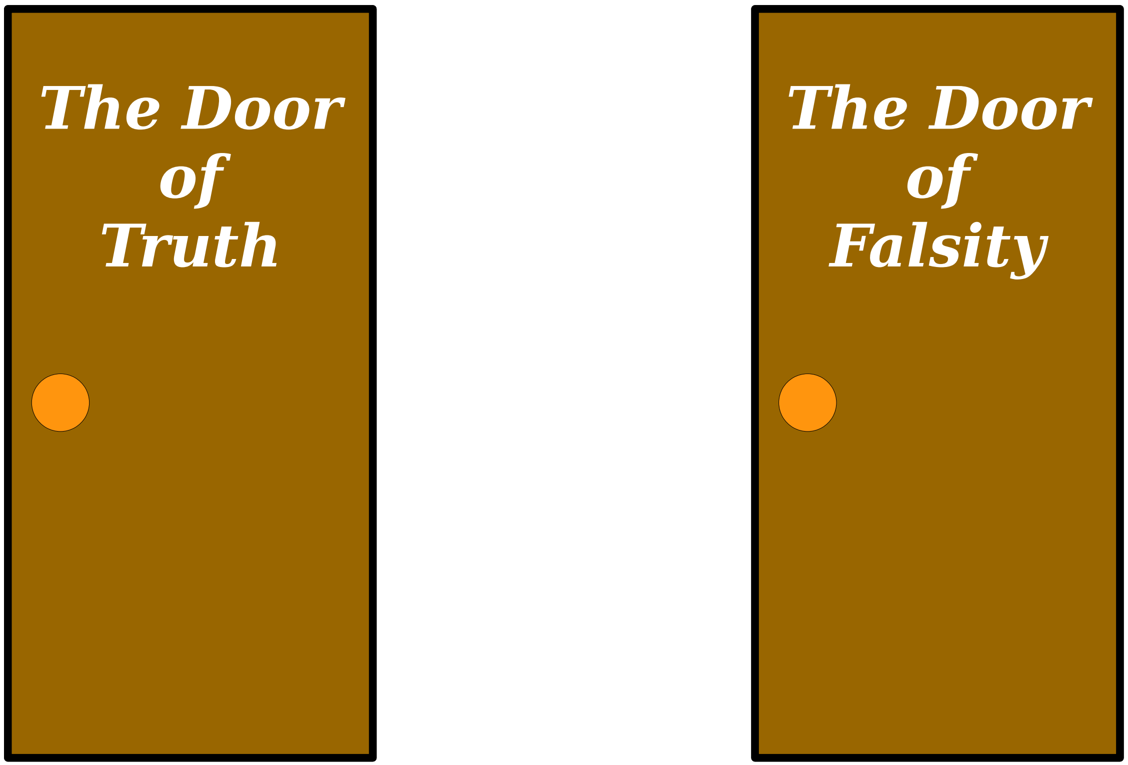


Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?



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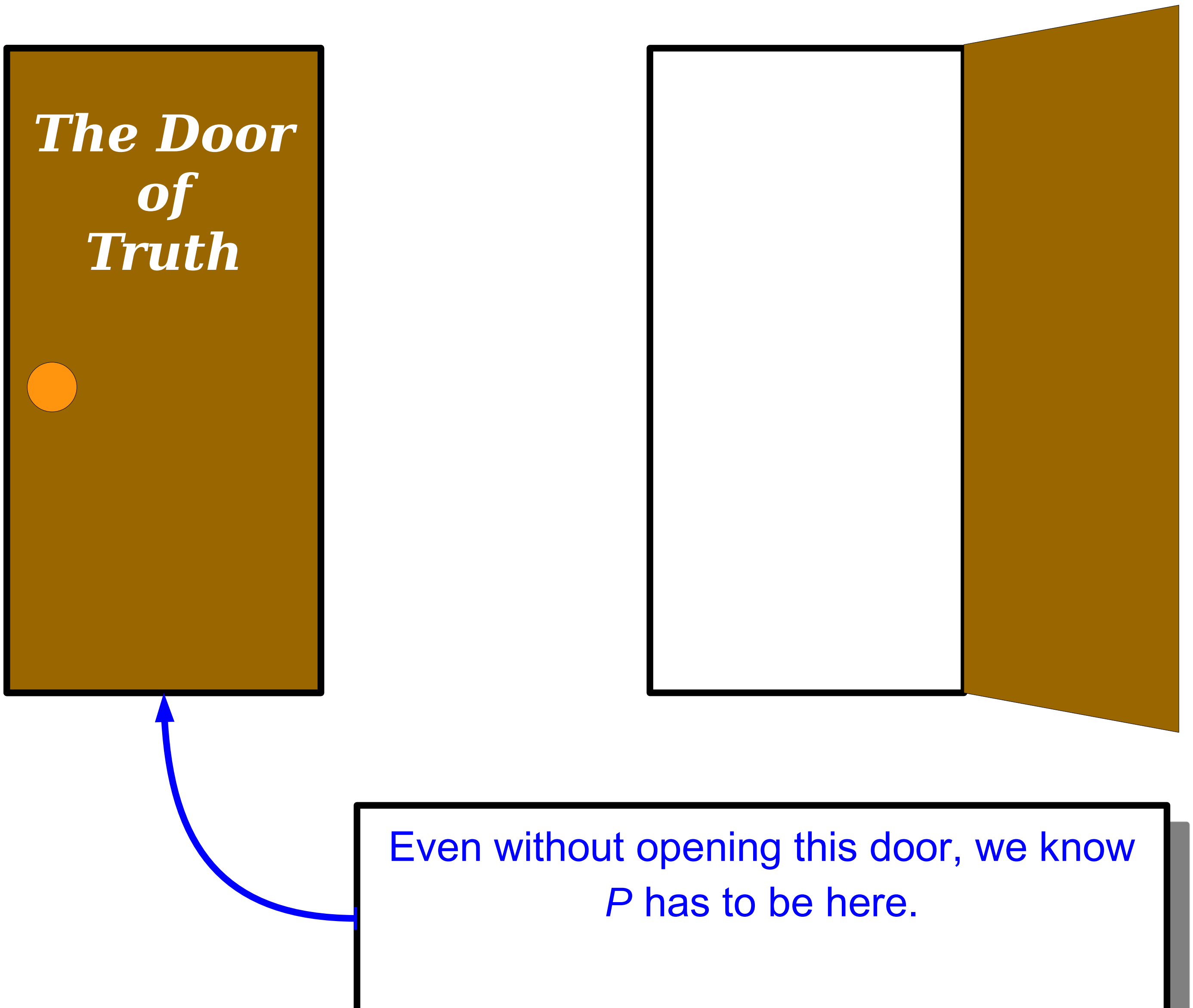

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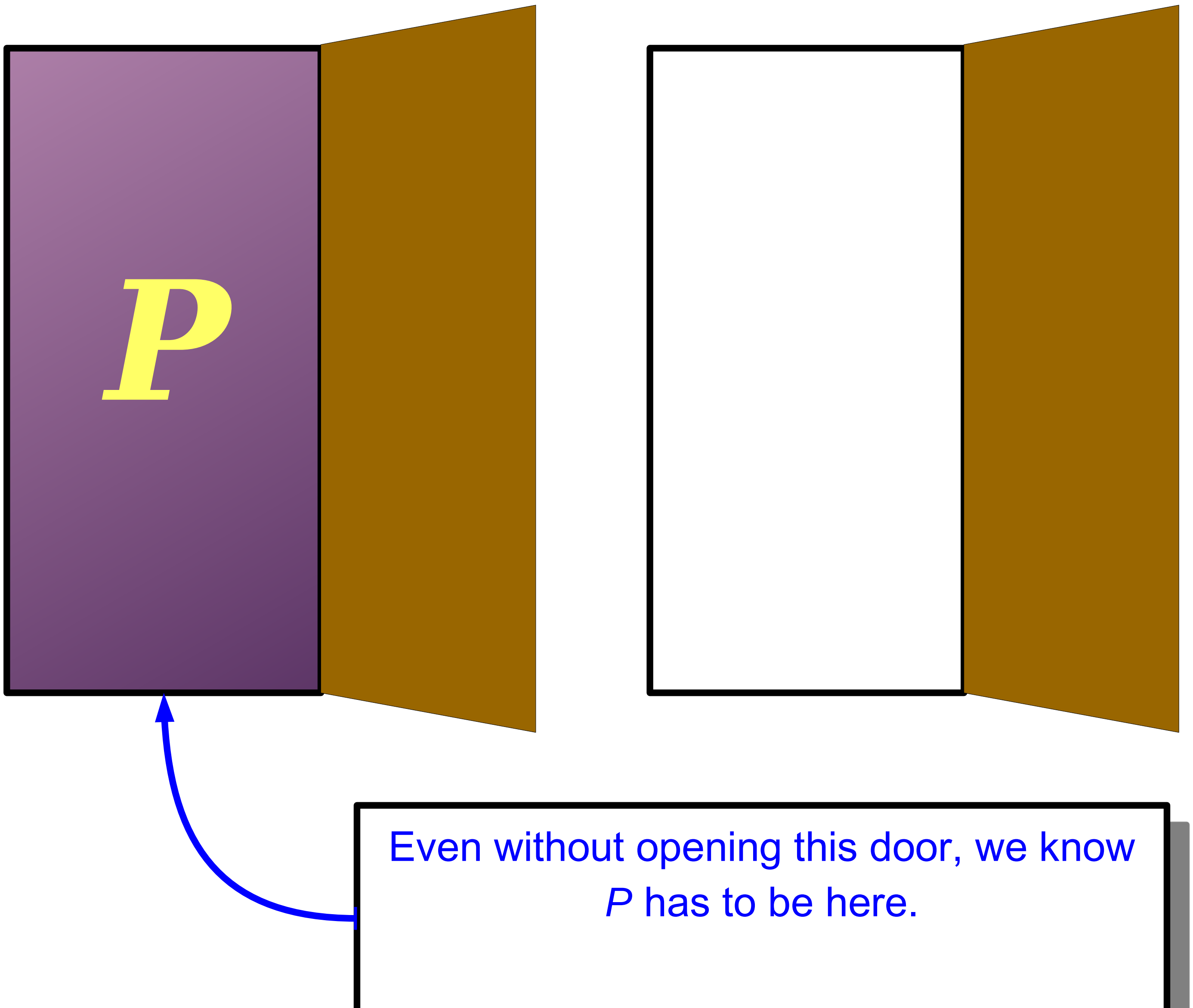

A ***proof by contradiction*** shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\varnothing| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

Theorem: There is no largest set.

Proof:

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

Theorem: There is no largest set.

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To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

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Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Notice that we're announcing

1. **that this is a proof by contradiction**, **and**
2. **what, specifically, we're assuming**.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

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Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong.

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We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set.

Theorem: There is no largest set.

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3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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    … what does this look like?

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Question: What is the negation of this statement?

Respond at pollev.com/robynreiss

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Question: How do we complete this sentence?

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Since n is odd we know that there is an integer k such that

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$$n^2 = (2k + 1)^2$$

Question: What would the rest of this proof look like? Remember, we are trying to arrive at some sort of contradiction. What can we say about n^2?

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    Assume ***P is true*** and ***Q is false***, then derive a contradiction.

What We Learned

- ***What's an implication?***
  - It's statement of the form "if *P*, then *Q*," and states that if *P* is true, then *Q* is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if *P*, then *Q*" is "if not *Q*, then not *P*.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement *P* that works by showing that *P* cannot be false.

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.